INTRODUCTION TO BANACH ALGEBRAS AND THE GELFAND-NAIMARK THEOREMS

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PREFACE

In these notes we give an introduction to the basic theory of Banach algebras, starting with a brief historical account of its development. C^*-algebras are studied in order to prove the commutative and the general Gelfand-Naimark theorem.
The material studied in these notes is mainly the product of a seminar I organized on Banach algebras at the Mathematics department of AUTH, attended by undergraduates and my supervisor Chariklia Konstadilaki-Savopoulou. Special Subject II of my study program consisted of the written in Greek version of these notes and the corresponding seminar (sections 1-11). Special Subject III contained to sections 12-16, the main parts of which I presented to the seminar of the Mathematical Analysis Section of the department of Mathematics of the AUTH.
I would like to thank my supervisor for her encouragement and support.

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CONTENTS

1 A brief historical account

The theory of Banach algebras (BA) is an abstract mathematical theory which is the (sometimes unexpected) synthesis of many specific cases from different areas of mathematics.
BA are rooted in the early twentieth century, when abstract concepts and structures were introduced, transforming both the mathematical language and practice. In the 1930's general topology has been quite developed while functional analysis was evolved through the Hahn-Banach theorem[1]. The uniform bound theorem, the theorem of closed graph and the open mapping theorem, all of them are 1932-theorems of Banach, whose book, "*Théorie des opérations linéaires*" (1932), influenced deeply mathematical analysis of his era.
Before Gelfand[2], who is the founder of the theory of BA, there were some papers dealing with the study of an additional multiplication on a Banach space (Nagumo, Yosida, von Neumann and others) without developing though a general theory.
In his dissertation thesis (1939) Gelfand recognized the central role of maximal ideals and by using their properties he created the modern theory of BA. Later (1941) these results are found in [Gelfand 1941]. In [Gelfand, Naimark 1943] Gelfand and Naimark[3] proved the two major representation theorems, named after them, which form the main body of the theory of BA. Mazur's 1938 theorem ([Mazur 1938]) was an important contribution to Gelfand's theory.

We give a historical line which describes roughly the development of the theory of BA:

1918: Riesz provides for the first time the axioms for a space with a norm $\|.\|$.

1920: Banach's thesis, first abstract study of normed spaces.

1929: von Neumann studies additional structure on a normed space.

1932: Stone's "Linear transformations in Hilbert space and their applications to Analysis" is a major contribution to operator theory.

1932: Banach's book "*Théorie des opérations linéaires*".

1932: Wiener introduced the inequality

$$\|xy\| \leq \|x\|\|y\|$$

without studying further consequences of it.

1936: The notion of abstract Banach algebra arises through Nagumo's "linear metric ring" and Yosida's "metrical complete ring".

1938: Mazur's theorem: every complex division algebra with a norm is isomorphic to $\mathbb{C}$ and every real one is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or the quaternions.

1939: Gelfand's thesis: foundations of the theory of commutative BA.

end of 30's: The term normed ring is established in the Soviet school.

1941: publication of the results of Gelfand's thesis and proof of Wiener's lemma.

[1] Both Hahn (1927) and Banach (1929) had shown the Hahn-Banach theorem for real spaces.
[2] Israil Moiseevich Gelfand (1913-) is the leading mathematician of the Russian school of the 20th century.
[3] Mark Aronovich Naimark 1909-1978.

1943: Gelfand-Naimark representation theorems.

1945: Ambrose introduces the term Banach algebra.

1947: Segal proves the real analogue to the commutative Gelfand-Naimark representation theorem.

1956: Naimark's book "*Normed Rings*" is the first presentation of the whole new theory of BA, which was important to its development.

1960: Rickart's book "*General theory of Banach algebras*" is the reference book of all later studies of BA.

The new theory of BA was a remarkable new general theory since it unified up to then distant areas of mathematics, providing new connections between functional analysis and classical analysis. The proof of Wiener's lemma was characteristic of the power of that new theory. Gelfand proved Wiener's lemma[4] (if f is non-zero and has an absolutely convergent Fourier expansion, then $\frac{1}{f}$ has such an expansion as well) in a few lines, attracting that way the attention of the mathematical community.
The Gelfand-Naimark theorems mainly influenced the spectral theory of operators on a Hilbert space. Actually, the commutative Gelfand-Naimark theorem is the spectral theorem of normal operators. Also, the constructions found in their proofs influenced the proofs of theorems from completely different areas (e.g., ?????).
In some sense the theory of BA is a generalization of the complex numbers. As we shall see this theory is an evolved way to talk about the algebra $\mathbb{C}$.
The theory of BA is used in theoretical physics, since C^*-algebras and von Neumann algebras are used in statistical mechanics and quantum mechanics. The real analogue to the commutative Gelfand-Naimark theorem was given by Segal (in [Segal 1947a]) in a paper on the axioms of quantum mechanics. Of course, it was few years before that quantum mechanics shaped the development of Hilbert spaces through the seminal book of von Neumann on the mathematical foundations of quantum mechanics [von Neumann 1932].
Finally, BA gave rise to the study of algebraic structures, quite analogue to algebras, with an additional compatible topological structure (topological algebras).
In the 1970's C^*-algebras were revitalized by the introduction of topological methods by Brown, Douglas and Fillmore on extensions of C^*-algebras, Elliott's use of K-theory to classify approximately finite dimensional C^*-algebras and Kasparov's melding of the two in KK-theory[5].

[4] See section...

[5] For these developments see [Davidson 1996]. For some recent interactions of the theory of BA with topics ranged from K-theory, over abstract harmonic analysis, to operator theory see [Lau, Runde 2004].

2 The Gelfand-Naimark theorems

We present here, roughly, the 1943-Gelfand-Naimark theorems.

Commutative Gelfand-Naimark theorem: If A is a commutative Banach algebra with involution, such that

$$\|x^*x\| = \|x^*\|\ \|x\|, \qquad \forall x \in A$$

then A is isometrically $*$-isomorphic to $C_0(X)$, the algebra of all continuous functions $X \to \mathbb{C}$ which vanish at infinity[6], where X is a locally compact Hausdorff space.

General Gelfand-Naimark theorem: If A is a Banach algebra with involution, such that

$$\|x^*x\| = \|x^*\|\ \|x\|, \qquad \forall x \in A$$

then A is isometrically $*$-isomorphic to a closed (with respect to the norm topology) $*$-subalgebra of $\mathfrak{B}(H)$, the bounded operators of some Hilbert space H.

The theorem that we shall prove here is the following version of the commutative case:

Commutative Gelfand-Naimark theorem$'$: A complex Banach algebra A is isometrically isomorphic to the algebra $\mathfrak{C}(K, \mathbb{C})$, of the continuous functions $K \to \mathbb{C}$, for some compact Hausdorff space K, if and only if it is commutative and there is an involution defined on A which turns it to a C^*-algebra.

The real analogue to the above theorem is Segal's theorem:

Real commutative Gelfand-Naimark theorem: A real Banach algebra A is isometrically isomorphic to the algebra $\mathfrak{C}(K, \mathbb{R})$, of the continuous functions $K \to \mathbb{R}$, for some compact Hausdorff space K, if and only if it is commutative and

(i) $\forall a \in A, \quad a^2 + 1$ has an inverse.

(ii) $\forall a \in A, \quad \|a^2\| = \|a\|^2$.

[6]If X is a locally compact Hausdorff space, a function $f : X \to \mathbb{C}$ vanishes at infinity iff $\forall \epsilon > 0\ \ \exists K$ compact subset of X such that $|f(x)| < \epsilon\ \ \forall x \in X_\infty - K$, where $X_\infty - K$ is a neighborhood of ∞ in the one point compactification X_∞ of X.

3 Basic definitions and comments

A vector space $\mathcal{A}$ over a field F is called an *algebra* (over F) iff
(i) there is a multiplication so that A becomes also a ring. The axiom of compatibility of the two structures is

$$\lambda(xy) = (\lambda x)y = x(\lambda y)$$

for every $\lambda \in F$.
(ii) the multiplication is associative.
(iii) there is a unit (which is preserved under homomorphisms).
(iv) F is $\mathbb{R}$ or $\mathbb{C}$.
So, $\mathcal{A} = (A, +, (F, \cdot), \cdot, 1)$.
A $*$-*algebra* $\mathcal{A}$ *over* $\mathbb{C}$ is a pair $(\mathcal{A}, *)$, where $\mathcal{A}$ is a complex algebra and an *involution* $* : A \to A$ which satisfies:

(i) $(x+y)^* = x^* + y^*$
(ii) $(\lambda x)^* = \bar{\lambda} x^*$, where $\bar{\lambda}$ is the conjugate of λ.
(iii) $(xy)^* = y^* x^*$
(iv) $x^{**} = x$

A subalgebra $\mathcal{B}$ of $\mathcal{A}$ is called $*$-*subalgebra* iff it is $*$-closed ($x \in B \to x^* \in B$). A $*$-*ideal* of $\mathcal{A}$ is an ideal of $\mathcal{A}$ satisfying the same condition. A $*$-homomorphism φ between $*$-algebras is an algebra homomorphism such that $\varphi(x^*) = \varphi(x)^*$. A $*$-*algebra* $\mathcal{A}$ *over* $\mathbb{R}$ is defined in a similar way.

An *algebra with a norm* $\mathcal{A}$ (over F) is a vector space with a norm such that $\mathcal{A}$ is also an F-algebra. The compatibility axioms of the two structures are:

(i) $\|xy\| \leq \|x\| \|y\|$
(ii)[7] $\|1\| = 1$

If $(A, \|.\|)$ is also a Banach space, then $\mathcal{A}$ is called a *Banach algebra*. An involution is called *isometrical* iff $\|x^*\| = \|x\|$. A *normed* $*$-*algebra* is an algebra with a norm, which is also a $*$-algebra. We do not set here compatibility axioms. A *Banach* $*$-*algebra* is a normed $*$-algebra which is a Banach algebra. A $*$-*Banach algebra* is a Banach $*$-algebra in which $*$ is isometrical.
A B^*-*algebra* is a Banach $*$-algebra satisfying the following property (Rickart 1946):

$$\|x^* x\| = \|x\|^2$$

Two normed $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ are *isometrically* $*$-*isomorphic* iff there is a $*$-isomorphism $\theta : A \to B$ which is also an isometry ($\|\theta(x)\| = \|x\|$). So θ is an isomorphism with respect to both the $*$-structure and the $\|.\|$-structure.

A normed $*$-algebra $\mathcal{A}$ satisfies the C^*-*condition* iff

$$\|x^* x\| = \|x^*\| \|x\| \ , \qquad \forall x \in A$$

A C^*-*algebra* is a Banach $*$-algebra satisfying the C^*-condition[8]. A fundamental example of a C^*-algebra is $\mathfrak{B}(H)$. A $\|.\|$-closed $*$-subalgebra of $\mathfrak{B}(H)$ is called a *concrete*

[7] Sometimes this property is excluded and a normed algebra satisfying it is called *unital*.
[8] The term C^*-algebra was introduced by Segal in [Segal 1947b].

C^*-*algebra*[9]. A *von Neumann algebra* (or a W^*-*algebra*) is a $*$-subalgebra of $\mathfrak{B}(H)$ which contains the identity operator and it is closed with respect to the weak operator topology (WOT)[10].

Comment 1: We have defined an algebra to be a vector space which is also a ring and not a ring which is also a vector space since
(i) there are non-associative algebras, while the multiplication is always associative in a ring.
(ii) as we saw in the historical development of the theory of BA it was the ring structure (i.e., the multiplication) that was added on the structure of the vector space.

Comment 2: An ideal $\mathcal{I} \subseteq A$ of $\mathcal{A}$ is a non-empty subspace of $\mathcal{A}$ (as a vector space) and an ideal of the ring $\mathcal{A}$. Trivially, $\mathcal{A}/\mathcal{I}$ is an algebra, the *quotient algebra* of $\mathcal{A}$ and $\mathcal{I}$.
So an ideal of the algebra $\mathcal{A}$ is also an ideal of the ring $\mathcal{A}$. The inverse holds if $\mathcal{A}$ has a unit, since if $\mathcal{I}$ is an ideal of the ring $\mathcal{A}$ it is also closed with respect to the scalar multiplication.

$$1 \in A \to \lambda i = \lambda(1i) = (\lambda 1)i \in \mathcal{I}$$

Comment 3: Since an algebra $\mathcal{A}$ is a special group, the representation theorem of Cayley (every group is embedded to its group of transformations) applies to $\mathcal{A}$. If $1 \in A$, the function $L : A \to L(A)$,

$$x \mapsto L_x \qquad L_x(y) = xy$$

is an embedding of $\mathcal{A}$ into the algebra $L(A)$ of linear transformations of $\mathcal{A}$. L is called the (left) *canonical representation* of $\mathcal{A}$. We shall refer to the canonical representation of $\mathcal{A}$ later.

Comment 4: The hypothesis that all algebras have a unit is harmless, since we can construct a unit as in the case of a ring without a unit. Namely, if $\mathcal{A}$ is an algebra without a unit, we consider $A_1 = A \times F$ which is a vector space as a product of vector spaces and its multiplication is defined by

$$(a, \lambda)\ (b, \mu) = (ab + \mu a + \lambda b, \lambda\mu)$$

This multiplication follows supposing (a, λ) and (b, μ) to be like $a + \lambda 1$, $b + \mu 1$ and multiplying as usual. $\mathcal{A}_1$ is obviously an algebra with unit the pair $(0, 1)$ and A is identified with $A \times \{0\}$. In this sense, it is easy to see that A is a maximal ideal of $\mathcal{A}_1$[11].
If $\mathcal{A}$ is also a normed space such that $\|xy\| \leq \|x\|\|y\|$, we define the following norm on $\mathcal{A}_1$:

$$\|(a, \lambda)\|_1 = \|a\| + |\lambda|$$

The identification between A and $A \times \{0\}$ becomes then an isometry ($\|(a, 0)\|_1 = \|a\| + |0| = \|a\|$). It is also easy to see that $\|.\|_1$ produces the product topology on

[9]The $\|.\|$-closed part of the definition explains the C symbol of a C^*-algebra.

[10]WOT is the weakest topology under which the functions $T \to (Tx, y)$ are continuous, $T \in \mathfrak{B}(H)$. von Neumann used the term *rings of operators* for them.

[11]Let M an ideal which strictly contains $A \times \{0\}$ and $(a, \lambda) \in M - (A \times \{0\})$. Then $\lambda^{-1}(a, \lambda) \in M - (A \times \{0\})$, therefore $\exists\ (b, 1) \in M - (A \times \{0\})$. Since $(-b, 0) \in A \times \{0\}$, $(b, 1) + (-b, 0) \in M$, but $(b, 1) + (-b, 0) = (0, 1)$, i.e., the unit of the algebra. Then $M = A_1$.

$A_1 = A \times F$. But since product topology is the topology of pointwise convergence it is trivial to check the following equivalence:

$$(a_n, \lambda_n) \stackrel{\|.\|_1}{\longrightarrow} (a, \lambda) \Leftrightarrow a_n \stackrel{\|.\|}{\longrightarrow} a \ \wedge \ \lambda_n \stackrel{|.|}{\longrightarrow} \lambda$$

So, $\mathcal{A}_1$ is a Banach space iff $\mathcal{A}$ is a Banach space. Also, $\mathcal{A}_1$ is commutative iff $\mathcal{A}$ is commutative.
$\|.\|_1$ is not the only one appropriate norm. Every norm which (i) turns $\mathcal{A}_1$ into a Banach space, (ii) gives the norm of $\mathcal{A}$ if it is restricted to $\mathcal{A}$ and (iii) defines the product topology, is a right one.
Of course, the usual procedure of completion of a normed space applies to a normed algebra leading to a Banach algebra.

Comment 5: $\mathbb{C}$ over $\mathbb{C}$ is the simplest complex Banach algebra, while $\mathbb{C}$ and $\mathbb{R}$ over $\mathbb{R}$ are the simplest real Banach algebras.
In bibliography usually only complex Banach algebras are defined. The general impression becomes then that Banach algebras are essentially complex objects i.e., that many results concerning complex Banach algebras do not have a real analogue. Quite often though, there is a real analogue, with a different formulation.
The relation between real and complex Banach algebras is not like the relation between real and complex Banach spaces, where the same facts hold, but it is close to the relation between the fields $\mathbb{R}$ and $\mathbb{C}$. The two major differences between $\mathbb{R}$ and $\mathbb{C}$, namely:

(i) $\mathbb{R}$ is not algebraically closed, while $\mathbb{C}$ is its algebraic closure, and
(ii) $\mathbb{R}$ has a complete order, while $\mathbb{C}$ cannot be an ordered field,

influence the relation between real and complex Banach algebras.

Comment 6: The involution mapping on a complex algebra is obviously the abstract version of the conjugate function $z \mapsto \overline{z}$ on complex numbers. This is the first indication of what we said earlier about the theory of BA being a general theory about $\mathbb{C}$. We shall encounter many other indications of this in the course of these notes, but at the same time we shall find differences between them too. It is trivial to see that

(i) $* : A \to A$ is 1-1 and onto A
(ii) $0^* = 0$
(iii) $1^* = 1$

Having $\mathbb{C}$ as our model we define the following kinds of elements of an abstract $*$-algebra:

(i) x is called *self-adjoint* or *hermitian* iff $x^* = x$.
(ii) x is called *normal* iff $x^*x = xx^*$.
(iii) x is called a *projection* iff $x^* = x$ and $x^2 = x$.
(iv) x is called *unitary* iff $x^*x = xx^* = 1$.
(v) x is called *anti-hermitian* iff $x^* = -x$.

Of course these concepts have their origin to the corresponding kinds of bounded operators of a Hilbert space H. But even the elements of $\mathbb{C}$ can be seen as such operators, as we have indicated in Comment 3.
If we use the symbols $Her(\mathcal{A}), Nor(\mathcal{A}), Pr(\mathcal{A}), Un(\mathcal{A}), Her^-(\mathcal{A})$ for the sets of the elements of $\mathcal{A}$ satisfying (i)-(v), then, obviously, all of them are subsets of $Nor(\mathcal{A})$.
The elements of $\mathbb{C}$ are all normal, but this is not the case in an abstract $*$-algebra.

In analogy to the following well known facts about complex numbers:

(i) $z = Rez + Imz$
(ii) $Rez = \frac{z+\overline{z}}{2}$ and $Imz = \frac{z-\overline{z}}{2i}$
(iii) $Rez \in Her(\mathbb{C})$ and $Imz \in Her^{-}(\mathbb{C})$

we can write an element of an abstract $*$-algebra as the following sum

$$x = \frac{x + x^*}{2} + \frac{x - x^*}{2}$$

where $\frac{x+x^*}{2} \in Her(\mathcal{A})$ and $\frac{x-x^*}{2} \in Her^{-}(\mathcal{A})$. If $\mathcal{A}$ was a complex $*$-algebra then the above analysis can be written:

$$x = \frac{x + x^*}{2} + \frac{x - x^*}{2i} i$$

since $char\mathbb{C} \neq 2$, therefore $\frac{1}{2}$ is meaningful. As in the case of $\mathbb{C}$ the above analysis is unique. To show this it suffices to show that 0 has a unique analysis. If $0 = a + b$ where $a^* = a$ and $b^* = -b$, then $a = -b$. But $a^* = (-b)^* = -b^* = -(-b) = b = -a$, which is absurd. We used the property

$$(-x)^* = -x^*$$

which holds, since $(x - x)^* = (x + (-x))^* = x^* + (-x)^* = 0^* = 0$.
We also note that the product of two elements $x = a + b$ and $y = c + d$, where $a, c \in Her(\mathcal{A})$ and $b, d \in Her^{-}(\mathcal{A})$ cannot be written as such a sum with respect to a, b, c, d unless the $*$-algebra is commutative (like $\mathbb{C}$).
Since the self-adjoint elements of $\mathbb{C}$ are the elements of $\mathbb{R}$, the self-adjoint elements of an $*$-algebra is the "$\mathbb{R}$-part" of the algebra.
Since

$$Her(\mathcal{A}) \cap Her^{-}(\mathcal{A}) = \{0\} \qquad (1)$$

and the above analysis is unique, then

$$A = Her(\mathcal{A}) \oplus Her^{-}(\mathcal{A})$$

In general, $Nor(\mathcal{A})$ is not a subalgebra of $\mathcal{A}$, though it is $*$-closed and $x \in Nor(\mathcal{A}) \rightarrow \lambda x \in Nor(\mathcal{A})$. It is trivial to see that:

(i) for two normal elements their sum and product are normal iff each one commutes with the conjugate of the other, and
(ii) an element of $\mathcal{A}$ is normal iff its real ($\frac{x+x^*}{2}$) and complex part ($\frac{x-x^*}{2i} i$) commute.
(iii) $Nor(\mathcal{A})$ is subalgebra of $\mathcal{A}$ iff $Nor(\mathcal{A}) = \mathcal{A}$[12]

Unitary elements of $\mathcal{A}$ generalize the most important subset of $\mathbb{C}$ after $\mathbb{R}$, the unit circle (the complex numbers $\|z\| = 1$). The elements of the unit circle satisfy

$$\|z\| = 1 \leftrightarrow z\overline{z} = \overline{z}z = 1$$

If $\mathcal{A}$ is a B^*-algebra and x is unitary element of $\mathcal{A}$ then

$$xx^* = x^*x = 1 \rightarrow \|z\| = 1$$

[12] We use (1) and the fact that $Her(\mathcal{A})$ and $Her^{-}(\mathcal{A})$ are subsets of $Nor(\mathcal{A})$.

since $\|xx^*\| = \|x\|^2 = 1$, therefore $\|x\| = 1$. Generally, the inverse does not hold.
So, in the theory of BA, which we consider as an abstract version of $\mathbb{C}$ we find that Banach algebras do not behave like $\mathbb{C}$ all the time.
This relation of closeness and distance between the theory of BA and $\mathbb{C}$ is shown by the following example:

If T is a bounded operator $T : H \to H$ on a Hilbert space H, then

T is unitary iff T is an isometrical isomorphism of H onto itself.

Let $\mathcal{A}$ be a B^*-algebra and u a unitary element of $\mathcal{A}$, so that $\|u\| = 1$. If we consider again the mapping

$$u \mapsto L_u \qquad L_u(x) = xu$$

we see that

$$\|L_u(x)\| = \|xu\| \leq \|u\| \; \|x\| = \|x\|$$

Since $L_{u^*} \circ L_u = L_{u^*u} = L_1$ and $L_1(x) = x$, then

$$\|x\| = \|L_1(x)\| = \|L_{u^*} \circ L_u\| = \|L_{u^*}(L_u(x))\| \leq \|L_{u^*}\| \; \|L_u(x)\| = \|L_u(x)\|$$

since, as we shall see later

$$\|L_x\| = \|x\|$$

therefore,

$$\|L_{u^*}\| = \|u^*\| = 1$$

So, we proved that

$$\|L_u(x)\| = \|x\|$$

i.e., L_u *is an isometry whenever* u *is unitary.*
Also, L_u *is an isomorphism*, since

(i) L_u is 1-1, since it is an isometry.
(ii) L_u is onto A, since $L_u(u^*x = u(u^*x) = x$.

The inverse though, does not hold i.e.,

L_u may be an isometrical isomorphism without u being a unitary element.

If $\mathcal{A}$ is a B^*-algebra and L_u is an isometrical isomorphism of $\mathcal{A}$ onto itself, then L_u is not unitary ($L_u^* = L_{u^*}$), since, generally, the norm of $\mathcal{A}$ is not generated by an inner product, as in the $\mathfrak{B}(H)$-case.
We see that Banach algebras is an evolved way to talk about $\mathbb{C}$ or $\mathfrak{B}(H)$, without being though, a simple generalization.

Comment 7: The mapping $x \mapsto L_x$ is an isometrical isomorphism of $\mathcal{A}$ on a Banach subalgebra of $\mathfrak{B}(\mathcal{A})$. the bounded operators on $\mathcal{A}$ since,

$$\|L_x(y)\| = \|xy\| \leq \|x\| \; \|y\|$$

therefore L_x is bounded and $\|L_x\| \leq \|x\|$. Also, since $L_x(1) = x1 = x$ and $\|1\| = 1$ we take the inverse inequality

$$\|L_x(1)\| = \|x\| \leq sup\{\|L_x(y)\|| \; \|y\| = 1\} = \|L_x\|$$

In that way we may identify the abstract Banach algebra $\mathcal{A}$ with a concrete Banach algebra of operators on $\mathcal{A}$. This is a first but quite trivial representation theorem for $\mathcal{A}$ It is trivial since $\mathfrak{B}(\mathcal{A})$ is also an abstract Banach algebra, which does not give information on $\mathcal{A}$. If we compare this representation with the Gelfand-Naimark representation we shall see the difference. In the commutative case we enter into $\mathfrak{C}(K, \mathbb{C})$, where K is a compact Hausdorff space, while in the general case we enter into $\mathfrak{B}(H)$ for some Hilbert space H. Of course, in order to do this it is necessary to represent only C^*-algebras. The L-representation is not an essential representation the way the Gelfand-Naimark representation is.

Comment 8: Property $\|xy\| \leq \|x\|\|y\|$ of a normed algebra guarantees that multiplication is jointly continuous with respect to x, y, namely

$$(x_n \to x \ \wedge \ y_n \to y) \Rightarrow x_n y_n \to xy$$

since

$$\|x_n y_n - xy\| = \|x_n y_n + x_n y - x_n y - xy\| = \|x_n(y_n - y) + (x_n - x)y\|$$

$$\leq \|x_n(y_n - y)\| + \|(x_n - x)y\| = \|x_n\| \ \|y_n - y\| + \|x_n - x\| \ \|y\|$$

since $\|x_n\|$ is bounded.
If the involution is isometric, then $\|x^*\| = \|x\|$ guarantees the continuity of the involution $*$. Namely, we wish to show

$$x_n \to x \Rightarrow x_n^* \to x^*$$

which is obvious, since

$$\|x_n^* - x^*\| = \|(x_n - x)^*\| = \|x_n - x\|$$

Moreover, in that case $*$ is a homeomorphism since $*^{-1}$ is $*$ itself.

Comment 9: In the next proposition we show that a B^*-algebra is a special kind of C^*-algebra.

Proposition 3.1: If $\mathcal{A}$ is B^*-algebra, then the elements of $\mathcal{A}$ satisfy the following properties:

(i) $\|x^*\| = \|x\|$
(ii) $\|x^* x\| = \|x^*\|\|x\|$ (i.e., the C^*-condition)
(iii) if x is a normal element of $\mathcal{A}$, then $\|x^2\| = \|x\|^2$

Proof: (i) Using the B^*-condition and the normed algebra inequality we get $\|x\|^2 = \|x^* x\| \leq \|x^*\| \ \|x\|$, so

$$\|x\| \leq \|x^*\| \leq \|x^{**}\| = \|x\|$$

(ii) Because of (i), the B^*-condition $\|x^* x\| = \|x\|^2$ becomes $\|x^* x\| = \|x^*\| \ \|x\|$, which is exactly the C^*-condition.
(iii) From the normed algebra condition we get $\|x^2\| \leq \|x\|^2$. For the inverse inequality we use the B^*-condition :

$$\|x^*\|^2 \|x\|^2 = (\|x^*\|\|x\|)^2 = \|x^* x\|^2 = \|(x^* x)^* x^* x\|$$

$$= \|x^*xx^*x\| = \|x^*x^*xx\| = \|(x^2)^*x^2\| = \|(x^2)^*\| \ \|x^2\|$$
$$= \|(x^*)^2\| \ \|x^2\| \leq \|x^*\|^2\|x^2\|$$

Therefore $\|x\|^2 \leq \|x^2\|$.

We are now in position to put more clearly the relation between $\mathbb{R}$ and $Her(\mathcal{A})$ and to give even simpler proofs to the generalizations of the following two propositions on the operators of a Hilbert space:

(H1) The self-adjoint operators of $\mathfrak{B}(H)$ is a closed real subspace of $\mathfrak{B}(H)$, therefore it is a real Banach space, which contains the identity.
(H2) The set of normal operators of $\mathfrak{B}(H)$ is a closed subset of $\mathfrak{B}(H)$ (which contains $Her(\mathcal{A})$).

Proposition 3.2: If $\mathcal{A}$ is normed $*$-algebra such that $\|x^*\| = \|x\|$ (or with a continuous involution), then:

($\mathcal{A}$1) $Her(\mathcal{A})$ is a closed $\mathbb{R}$-subspace of $\mathcal{A}$. So, if $\mathcal{A}$ is a Banach algebra (like $\mathfrak{B}(H)$), then $Her(\mathcal{A})$ is a real Banach space.
($\mathcal{A}$2) $Nor(\mathcal{A})$ is a closed subset of $\mathcal{A}$.

Proof: (i) If $x_n \in Her(\mathcal{A})$ and $x_n \to x$, then $x_n^* \to x^*$ and since $x_n = x_n$, then $x^* = x$.
(ii) If $x_n \in Nor(\mathcal{A})$ and $x_n \to x$, then $x_n^* \to x^*$ and $x_n^*x_n \to x_nx_n^*$. Since multiplication is continuous $x_n^*x_n \to x^*x$ and $x_nx_n^* \to xx^*$ we get $x^*x = xx^*$.

Comment 10: The non-commutative Gelfand-Naimark theorem connects an abstract C^*-algebra to a concrete C^*-algebra. Every C^*-algebra is isomorphic (with respect to the whole of its structure) to a C^*-subalgebra of $\mathfrak{B}(H)$ for some Hilbert space H.

Note that if x_n a sequence in $\mathcal{A}$ which satisfies the C^*-condition, such that $x_n \overset{\|\cdot\|}{\to} x$, then x also satisfies the C^*-condition.

4 Examples

The following examples of Banach algebras belong to different areas of mathematical analysis, which are unified in the theory of BA. Generally, there are three kinds of examples depending on the way which the multiplication is defined.

(I) Operator Algebras: Multiplication is defined as composition. Unit 1 is always the identity operator I.
Let X be a normed space. The set $\mathfrak{B}(X)$ of linear and bounded operators T, where

$$\|T\| = sup\{\|Tx\| \mid x \in X \ \wedge \ \|x\| \leq 1\}$$

is a normed algebra since,

$$\|TSx\| \leq \|T\| \ \|Sx\| \leq \|T\| \ \|S\| \ \|x\|$$

$$\Rightarrow \|TS\| \leq \|T\| \ \|S\|$$

Also, since $\|Ix\| = \|x\| \leq \|x\|$, then $\|I\| \leq 1$ and since $\|x\| = \|Ix\| \leq \|I\| \ \|x\|$, then $1 \leq \|I\|$.
$\mathfrak{B}(X)$ is a Banach algebra iff X is a Banach space.
If $X = \mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then $\mathfrak{B}(X) = \mathbb{K}^{n \times n}$, since every linear mapping from $\mathbb{K}^n$ to $\mathbb{K}^n$ is continuous (i.e., bounded) and these mappings are isomorphic to the $n \times n$ matrices with $\mathbb{K}$-entries.

If H is a Hilbert space, then $\mathfrak{B}(H)$ is a Banach algebra. To show that $\mathfrak{B}(H)$ is an $*$-algebra we use the representation theorem of Riesz.

Riesz representation theorem: F is a bounded linear functional on a Hilbert space H ($F \in H^*$) iff there exists a unique vector z in H such that $F(x) = F_z(x) = <x, z>$, $x \in H$, where $<,>$ is the inner product in H. Also, $\|F\| = \|z\|$.

Let $T \in \mathfrak{B}(H)$. If we fix $y \in H$, then the linear mapping

$$x \mapsto <Tx, y>$$

is a functional on H. By Riesz representation theorem there is a unique z such that

$$<Tx, y> = <x, z>$$

. We define then $T^* : H \to H$, the adjoint of T, by $T^*y = z$ and the pervious equality becomes

$$<Tx, y> = <x, T^*y>,$$

the definitional equality of T^*. This natural definition of T^* cannot be given in an abstract Banach space the norm of which does not come from an inner product. $\mathfrak{B}(H)$ is C^*-algebra since the following hold:

Proposition 4.1: (i) $\|T^*\| = \|T\|$.
(ii) $\|T^*T\| = \|T\|^2$.

Proof: (i)

$$\|T^*x\|^2 = <T^*x, T^*x> = <TT^*x, x> \leq \|TT^*x\| \ \|x\|$$

(using the Cauchy-Schwarz inequality

$$| < x, y > | \leq \|x\| \ \|y\|$$

and the fact that $< TT^*x, x > = \|T^*x\|^2 \geq 0$). And since, $\|TT^*x\| \ \|x\| \leq \|T\| \ \|T^*\| \ \|x\|$ we get $\|T^*x\| \leq \|T\| \ \|x\|$ i.e, $\|T^*\| \leq \|T\|$. Using the $*$-property

$$T^{**} = T$$

and applying the last inequality we get

$$\|T^{**}\| = \|T\| \leq \|T^*\|.$$

(ii) From the normed algebra inequality and the (i) case

$$\|T^*T\| \leq \|T^*\| \ \|T\| = \|T\|^2.$$

The inverse inequality is proved as in the (i) case.

$$\|T^*x\|^2 = < T^*x, T^*x > = < TT^*x, x > \leq \|TT^*x\| \ \|x\| \leq \|TT^*\| \ \|x\|^2$$

Since $\|T^*x\|^2 \leq \|T^*\|^2 \ \|x\|^2$ we get $\|T^*\|^2 = \|T\|^2 \leq \|TT^*\|$.

(II) Function Algebras: Multiplication is defined pointwisely and the unit is "essentially" the constant function 1.

(A) If X is a non-empty set, then $l^\infty(X, F)$ is the set of all bounded functions $f : X \to F$, which becomes a Banach algebra with

$$\|f\| = sup\{|f(x)| \ |x \in X\}$$

We can see $l^\infty(X, F)$, or simpler $l^\infty(X)$, in a more general setting.
If $(X, \mathcal{F}, \mu)$ is a measure space and $f : X \to F$ is a measurable function, then we define the *essential supremum* of f by

$$\|f\|_\infty = inf\{r| \ r \in [0, +\infty] \ : \ \mu(\{x \in X \mid |f(x)| > r\}) = 0\}.$$

The term is justified by the fact that the set of the elements of X for which f is not bounded by $\|f\|_\infty$ is of zero measure. f is called essentially bounded iff $\|f\|_\infty < \infty$. Let $\mathcal{L}^\infty(\mu)$ be the set of essentially bounded measurable functions. Then, $\mathcal{L}^\infty(\mu)$ is an algebra with $\|f\|_\infty$ as a semi-norm, since $\|f\|_\infty = 0 \nrightarrow f = 0$ (just take a function being 0 everywhere, except a subset of measure 0 on which it is constant). Defining the equivalence relation $f \sim g \leftrightarrow f = g$ μ-a.e., the quotient $L^\infty(\mu) = \mathcal{L}^\infty(\mu)/\sim$ becomes a Banach algebra with $\|[f]\| = \|f\|_\infty$. It is easy to see that $\|[f]\|$ is well defined and it is actually a norm. That $L^\infty(\mu)$ is a Banach space is a special case of Riesz-Fischer theorem for $L^p(\mu)$ spaces, $1 \leq p \leq \infty$. On the other hand, the normed algebra properties are straightforward.
Unit is the class [1]. To show

$$\|fg\|_\infty \leq \|f\|_\infty \ \|g\|_\infty$$

we work as follows.

$$\|f\|_\infty = infF \qquad F = inf\{r_1| \ r_1 \in [0, +\infty] \ : \ \mu(\{x \in X \mid |f(x)| > r_1\}) = 0\}$$

$$\|g\|_\infty = infG \qquad G = inf\{r_2|\ r_2 \in [0,+\infty]\ :\ \mu(\{x \in X \mid |f(x)| > r_2\}) = 0\}$$
$$\|fg\|_\infty = infF\circ G \qquad F\circ G = inf\{r|\ r \in [0,+\infty]\ :\ \mu(\{x \in X \mid |f(x)g(x)| > r\}) = 0\}$$

Since F, G are subsets of positive reals $infF\ infG = infFG$. It suffices to show

$$r_1 \in F,\ r_2 \in G \Rightarrow r_1r_2 \in F\circ G \Rightarrow infF\circ G \leq infFG = infF\ infG.$$

If $r_1 \in F$, then $\mu(F_1) = 0$, where

$$F_1 = \{x \in X\ : |f(x)| > r_1\}.$$

If $r_2 \in G$, then $\mu(G_2) = 0$, where

$$G_2 = \{x \in X\ : |g(x)| > r_2\}.$$

In order r_1r_2 belongs to $F\circ G$ we must show that

$$\mu(\{x \in X \mid |f(x)g(x)| > r_1r_2\}) = 0.$$

Let $x \in X$ such that $|f(x)g(x)| > r_1r_2$. Necessarily then, x belongs either to F_1 or to G_2 (otherwise $|f(x)| < r_1$ and $|g(x)| < r_2$). So, $\{x \in X \mid |f(x)g(x)| > r_1r_2\} \subseteq F\cup G$, therefore it is of measure 0 and r_1r_2 belongs to $F\circ G$.

If $(X, \mathcal{F}, \mu) = (X, \mathcal{P}(X), \nu)$ where $\nu(A)$ is n, if A has n elements and $\nu(A)$ is ∞, if A is infinite, then

$$\mathcal{L}^\infty(\nu) = L^\infty(\nu) = l^\infty(X)$$

and the $L^\infty(\mu)$-case is a generalization of the $l^\infty(X)$-case. To see this let $f : X \to F$ be a measurable function. Then,

$$\|f\|_\infty = inf\{r|\ r \in [0,+\infty]\ :\ \nu(\{x \in X \mid |f(x)| > r\}) = 0\}.$$

But,

$$\nu(\{x \in X \mid |f(x)| > r\}) = 0 \Leftrightarrow \{x \in X \mid |f(x)| > r\} = \emptyset$$
$$\Leftrightarrow (\forall x)\ x \in X\ \ |f(x)| \leq r$$

So, when f is essentially bounded, then f is bounded. Also,

$$\|f\|_\infty = inf\{r|\ r \in [0,+\infty]\ :\ |f(x)| \leq r,\ \forall x \in X\} = sup\{|f(x)| \mid x \in X\}.$$

To show that $mathcalL^\infty(\nu) = L^\infty(\nu)$ it suffices to show that $[f] = \{f\}$. But

$$\nu(\{x \in X :\ f(x) \neq g(x)\}) = 0 \Leftrightarrow \{x \in X :\ f(x) \neq g(x)\} = \emptyset \Leftrightarrow f = g.$$

(B) Let X be a topological space. The space $C_b(X)$ of continuous and bounded functions $f : X \to \mathbb{K}$ is a closed subalgebra of $l^\infty(X)$, therefore it is a Banach algebra. As we shall show later $C_b(X)$ is actually $C(\beta X)$, where βX is the Stone-Čech compactification of X (without loss of generality X is considered to be compact Hausdorff space). If X is a singleton, then $C_b(X)$ is $\mathbb{C}$ (if $\mathbb{K}$ is $\mathbb{C}$).

(C) The space $C^1[a, b]$ of continuously differentiable functions $f : [a, b] \to \mathbb{K}$ is a Banach space, where

$$||f|| = ||f||_\infty + ||f'||_\infty.$$

It is also a Banach algebra since,

$$||fg|| = ||fg||_\infty + ||(fg)'||_\infty = ||fg||_\infty + ||fg' + f'g||_\infty \leq$$
$$||f||_\infty ||g||_\infty + ||fg'||_\infty + ||f'g||_\infty \leq$$
$$||f||_\infty ||g||_\infty + ||f||_\infty ||g'||_\infty + ||f'||_\infty ||g||_\infty \leq$$
$$||f||_\infty ||g||_\infty + ||f||_\infty ||g'||_\infty + ||f'||_\infty ||g||_\infty + ||f'||_\infty ||g'||_\infty = ||f|| ||g||.$$

Also,

$$||1|| = ||1||_\infty + ||1'||_\infty = ||1||_\infty = 1.$$

(D) Example of a Banach algebra without a unit: Let X is a locally compact, not compact, Hausdorff space and

$$C_0(X) = \{f |\ f \in C(X),\ f\ vanishes\ at\ infinity\},$$

where "*f vanishes at infinity*" means that $\forall \varepsilon > 0, \exists K,\ K$ is a compact subset of X such that, $|f(x)| < \varepsilon$, for each x in $X - K$. If X_∞ is the one-point compactification of X, then f vanishes at infinity iff $lim_{x \to \infty} f(x) = 0$. Thus,

$$C_0(X) = \{f |\ f \in C(X_\infty),\ f(\infty) = 0\}.$$

$C_0(X)$ is a closed subalgebra of $C(X_\infty)$, therefore a Banach algebra, without a unit. To show this we view $C_0(X)$ as a maximal ideal of $C(X_\infty)$. We know that if K is a compact Hausdorff space, then the sets

$$M_x = \{f |\ f \in C(K),\ f(x) = 0\}$$

are maximal ideals of $C(K)$. Hence,

$$C_0(X) = M_\infty$$

i.e., $C_0(X)$ is a maximal ideal of $C(X_\infty)$, which is an algebra with a unit, therefore, the ideal of the ring $C(X_\infty)$ is also an ideal of the algebra $C(X_\infty)$. Of course M_∞ has no unit, since $lim_{x \to \infty} 1(x) = 1 \neq 0$.
$C_0(X)$ is closed in $C(X_\infty)$, since, if $(f_n)_n$ is a sequence in $C_0(X)$ and $f_n \to f$ with respect to $||.||_\infty$, then $f_n \to f$ pointwisely. Thus, $f_n(\infty) \to f(\infty)$, therefore, $f(\infty) = 0$ [13].
The unitization of $C_0(X)$ is obviously $C(X_\infty)$ [14] and we get the "identification"

$$C(X_\infty) = \{f |\ f \in C(X),\ \exists lim_{x \to \infty} f(x)\}.$$

[13] We may prove that M_∞ is closed as follows: M_∞ is a maximal ideal of $C(X_\infty)$, which is a Banach algebra with a unit. The closure of M_∞ is also an ideal, of which we do not know if it is proper. It can be proved though, that if $\mathcal{A}$ is a Banach algebra with 1 and M is a proper ideal of $\mathcal{A}$, then $\overline{M}$ is also proper. Thus, $\overline{M}_\infty$ is proper and maximal, therefore, it is closed.

[14] $C(X_\infty)$ and $C_0(X) \times K$ are actually the "same". We define

$$e : C_0(X) \times K \to C(X_\infty),$$

where

$$(f, k) \mapsto \widetilde{f} + k1,$$

and $\widetilde{f} : X_\infty \to K$, such that $\widetilde{f}|_X = f$. e is a ring (and algebras) isomorphism such that

$$e(0, 1) = \widetilde{0} + 11 = 1$$

and the two norms are equivalent.

$\mathbb{N}$ is a locally compact, non-compact Hausdorff space, for which

$$C_0(\mathbb{N}) = \{f|\ f : \mathbb{N} \to K,\ lim_{n\to\infty} f(n) = 0\} = c_0,$$

and

$$C(\mathbb{N}_\infty) = \{f|\ f : \mathbb{N} \to K,\ \exists lim_{n\to\infty} f(n)\} = c.$$

Proposition 4.2: $C_0(X)$ has a unit such that $||1|| = 1$ iff X is compact.

Proof: Let $C_0(X)$ has such a unit. Since $C_0(X)$ is a maximal (proper) ideal of $C(X_\infty)$, having a unit makes it identical to $C(X_\infty)$, therefore $X = X_\infty$.
If X is compact, then taking $K = X$, $|f(x)| < \varepsilon$, for each x in $\emptyset = X - X$. Therefore, 1 is trivially in $C_0(X)$.

(E) If $\mathcal{A}$ is the algebra of all complex polynomials in $[0, 1]$ i.e.,

$$p(x) = \sum_{n=0}^{N} a_n x^n, \quad a_n \in \mathbb{C},\ x \in [0, 1],$$

with the sup norm, then $\mathcal{A}$ is a commutative normed algebra with a unit, which is *not* complete.

(F) Let $D = \{z|\ z \in \mathbb{C}\ :\ ||z|| \leq 1\}$. Then,

$$H(D) = \{f|\ f \in C(D)\ :\ f|_{D^o}\ \textit{is holomorphic}\}$$

is a subalgebra of $C(D)$. Combining Cauchy-Goursat and Morera's theorem we get that a uniform limit of holomorphic functions is holomorphic, therefore $H(D)$ is closed in $C(D)$, i.e., it is a Banach algebra, and it is called the *disc algebra*.

If in examples (II) (A)-(C) $\mathbb{K} = \mathbb{C}$, then an involution $f \mapsto f^*$ is defined, where $f^*(x) = \overline{f(x)}$. $H(D)$ is not closed under $*$, since $z \mapsto \overline{z}$ is not holomorphic. In $H(D)$ we define the following involution

$$f^*(z) = \overline{f(\overline{z})}.$$

Also, if $\mathbb{K} = \mathbb{C}$, algebras $L^\infty(\mu)$ and $C_b(X)$ are C^*-algebras, but $C^1[a, b]$ and $H(D)$ are not[15], satisfying though the weaker condition $||f^*|| = ||f||$.

Next algebra is on $\{0, 1\}$. Each Boolean algebra "is" a Boolean ring i.e., a ring with unit, such that $x^2 = x$, for each x. Multiplication is automatically commutative and by defining

$$||x|| = \begin{cases} 1 & \text{, if } x \neq 0 \\ 0 & \text{, if } x = 0 \end{cases}$$

and $x^* = x$, it becomes a C^*-algebra.

(III) Group Algebras: Multiplication is the operation of convolution. These algebras do not always have a unit.
(A) If $G = \{g_1, g_2, ..., g_n\}$ is a finite group, we define

$$L^1(G) = \{f|\ f : G \to \mathbb{C}\}.$$

[15]Consider for example $f(t) = t - a$ for $C^1[a, b]$, and $f(z) = z + i$ for $H(D)$.

Addition and multiplication are defined pointwisely. With the norm

$$||f|| = \sum_{i=1}^{n} |f(g_i)|$$

$L^1(G)$ becomes a Banach space.
The definition of convolution is understood if we see an element f of $L^1(G)$ as a sum

$$\sum_{i=1}^{n} a_i g_i, \quad a_i = f(g_i).$$

In that way we define

$$(\sum_{i=1}^{n} a_i g_i)(\sum_{j=1}^{n} b_j g_j) = \sum_{k=1}^{n} c_k g_k,$$

where

$$c_k = \sum_{g_i g_j = g_k} a_i b_j.$$

I.e., the coefficient of g_k, the value of convolution of two functions on g_k, is the sum of all products $a_i b_j$ for which $g_i g_j = g_k$. Thus, we define the convolution of f and g to be

$$(f * h)(g_k) = \sum_{g_i g_j = g_k} f(g_i) h(g_j).$$

Since $g_i g_j = g_k$, $g_i = g_k g_j^{-1}$, and $(f * h)(g_k)$ is written

$$(f * h)(g_k) = \sum_{j=1}^{n} f(g_k g_j^{-1}) h(g_j).$$

Convolution is defined so that G "enters" $L^1(G)$. Each element g_i of G can be seen as the following element of $L^1(G)$:

$$f(g_k) = \begin{cases} 0 & \text{, if } k \neq i \\ 1 & \text{, if } k = i \end{cases}$$

By that way $e : G \to L^1(G)$ is established, where $g_i \mapsto e(g_i)$ and $e(g_i)(g_k)$ is defined as above.
It is easy to see that

$$e(g_i g_j) = e(g_i) e(g_j)$$

and if g_{i_0} is the identity element of G, then $e(g_{i_0})$ is the unit of $L^1(G)$ with respect to $*$. Obviously, every element of G has norm 1 as an element of $L^1(G)$, therefore its unit has also norm 1. Also,

$$||f * h|| = \sum_{k=1}^{n} |(f * h)(g_k)| =$$

$$\sum_{i=1}^{n} | \sum_{g_i g_j = g_k} f(g_i) h(g_j)| \leq \sum_{k=1}^{n} \sum_{g_i g_j = g_k} |f(g_i) h(g_j)| \leq$$

$$\sum_{k=1}^{n} \sum_{g_i g_j = g_k} |f(g_i)||h(g_j)| \leq ||f||||h||.$$

Therefore, $L^1(G)$ is a Banach algebra.

(B) If $G = \mathbb{Z}$, then

$$L^1(\mathbb{Z}) = \{f| \ f : \mathbb{Z} \to \mathbb{C} \ : \ \sum_{k \in \mathbb{Z}} |f(k)| < \infty\}.$$

$L^1(\mathbb{Z})$ is a commutative Banach algebra with norm

$$\|f\| = \sum_{k \in \mathbb{Z}} |f(k)|$$

and multiplication the convolution operation

$$(f * g(n)) = \sum_{k \in \mathbb{Z}} f(n-k)g(k).$$

(C) If in $L^1(\lambda)$, the set of Lebesgue-integrable functions, we define the convolution to be

$$(f * g)(x) = \int f(x-y)g(y)d\lambda(y),$$

$\forall x \in \mathbb{R}$ for which the mapping

$$y \mapsto f(x-y)g(y)$$

is λ-integrable, and $f * g = 0$, for all the rest $x \in \mathbb{R}$, which are of Lebesgue measure 0, then $L^1(\lambda)$ becomes an algebra Banach.

(D) If G is a locally compact Hausdorff topological group and μ is a left Haar measure on G, then $L^1(G)$, the set of all classes of complex Borel-measurable functions on G, such that

$$\int_G |f| d\mu < \infty,$$

then $L^1(G)$ is a Banach space with operations pointwisely defined and with norm

$$||f|| = \int_G |f| d\mu.$$

With multiplication

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)d\mu(y),$$

it becomes a Banach algebra, known as "group algebra", which generally has no unit and it is not commutative. $L^1(G)$ is central to harmonic analysis.

5 Spectrum

We know that if $\lambda \in \mathbb{R}$ and $|\lambda| < 1$, then the geometric series

$$\sum_{n=0}^{\infty} \lambda^n$$

converges, and

$$\sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}.$$

The above fact is generalized in Banach algebras.

Proposition 5.1: If $\mathcal{A}$ is a Banach algebra and $a \in A$, such that $||a|| < 1$, then $1-a$ is invertible, the series

$$\sum_{n=0}^{\infty} a^n$$

converges, and

$$\sum_{n=0}^{\infty} a^n = (1-a)^{-1}.$$

Proof: Since

$$\sum_{n=0}^{\infty} ||a^n|| \leq \sum_{n=0}^{\infty} ||a||^n$$

and $||a|| < 1$ the series converges absolutely, and since A is a Banach space, it converges. Also,

$$(1-a)\sum_{n=0}^{\infty} a^n = \sum_{n=0}^{\infty} a^n - \sum_{n=1}^{\infty} a^n = a^0 = 1.$$

Similarly,

$$\sum_{n=0}^{\infty} a^n(1-a) = 1.$$

Therefore, the needed equality is proved.

Many of the following results rest on the previous fact, in which the completeness of a Banach space was crucial.

Proposition 5.2: If $\mathcal{A}$ is a Banach algebra and

$$U(A) = \{a|\ a \in A,\ a \textit{ is invertible}\},$$

then $U(A)$ is an open set of A.

Proof: Proposition 5.1 is equivalent to

$$||1-a|| < 1 \Rightarrow a \in U(A)$$

i.e., the ball of the unit with radius 1 is a subset of $U(A)$. Let $a_0 \in U(A)$ [16]. Consider the ball with center a_0 and radius $\frac{1}{||a_0^{-1}||}$. Then,

$$||a - a_0|| < \frac{1}{||a_0^{-1}||}$$

[16] $U(A)$ is non-empty, since invertibility presupposes the existence of a unit, which is trivially in $U(A)$.

implies that

$$||a_0^{-1}a - 1|| = ||a_0^{-1}(a - a_0)|| \leq ||a_0^{-1}||||a - a_0|| < 1.$$

Therefore, $a_0^{-1}a \in U(A)$. But then $a \in U(A)$, since $a = a_0(a_0^{-1}a)$ i.e., since a is written as the product of two invertible elements.

Proposition 5.3: If $\mathcal{A}$ is a normed algebra, then the operation $^{-1} : U(A) \to U(A)$, where $a \mapsto a^{-1}$, is continuous.

Proof: (i) Let $\mathcal{A}$ be a Banach algebra.
We show first that $^{-1}$ is continuous at 1.
If $\varepsilon > 0$, we need to find $\delta > 0$ such that $||a - 1|| < \delta \Rightarrow ||a^{-1} - 1|| < \varepsilon$. If $\delta \leq 1$, then

$$||a - 1|| < \delta \Rightarrow a^{-1} = \sum_{n=0}^{\infty}(1 - a)^n.$$

Therefore,

$$||a^{-1} - 1|| = ||\sum_{n=1}^{\infty}(1 - a)^n|| \leq \sum_{n=1}^{\infty}||(1 - a)^n|| \leq$$

$$\sum_{n=1}^{\infty}||(1 - a)||^n < \sum_{n=1}^{\infty}\delta^n = \frac{\delta}{1 - \delta} = \varepsilon,$$

if we choose $\delta = \frac{\varepsilon}{1+\varepsilon}$.
Let $a \neq 1$ in $U(A)$ and $a_n \to a$, where $a_n \in U(A)$. Then,

$$a_n a^{-1} \to 1 \Rightarrow (a_n a^{-1})^{-1} \to 1 \Leftrightarrow a a_n^{-1} \to 1 \Rightarrow$$

$$a^{-1}(a a_n^{-1}) \to a^{-1} \Leftrightarrow a_n^{-1} \to a^{-1}.$$

(ii) If $\mathcal{A}$ is a normed algebra, then $^{-1}$ is the restriction of $^{-1}$ on $U(\overline{A})$ to $U(A)$, where $\overline{A}$ is the completion $\mathcal{A}$. Therefore $^{-1}$ is continuous.
Also, since $^{-1}$ is identical to its inverse, it is also a homeomorphism of the topological group $U(A)$ onto itself.

If a is invertible in $\mathcal{A}$ does not entail that it is invertible in every subalgebra of $\mathcal{A}$. Likewise, a non-invertible element of a subalgebra of $\mathcal{A}$ may be invertible in $\mathcal{A}$. E.g., if $P[0, 1]$ is the algebra of polynomials on $[0, 1]$ with complex values, then $p(x) = x + 1$ is not invertible in $P[0, 1]$ but it is in $C[0, 1]$, which includes $P[0, 1]$. There are certain non-invertible elements of a normed algebra $\mathcal{A}$ though, which remain non-invertible in each normed algebra which contains $\mathcal{A}$. Such elements are the topological divisors of zero.
An element z of a normed algebra $\mathcal{A}$ is called a *topological divisor of zero* iff there is sequence (z_n) of elements of $\mathcal{A}$, such that $||z_n|| = 1$ and $z_n z \to 0$ or $z z_n \to 0$. Obviously, a divisor of zero is a topological divisor of zero. If

$$Z(A) = \{z |\ z \in A, z\ is\ a\ topological\ divisor\ of\ zero\}$$

and

$$S(A) = \{a |\ a \in A, a\ is\ not\ invertible\},$$

then

$$Z(A) \subseteq S(A).$$

To show this we suppose that a topological divisor of zero z is in $U(A)$. Since

$$z_n z \to 0 \Rightarrow (z_n z)z^{-1} \to 0 \Leftrightarrow z_n \to 0,$$

$||z_n||$ cannot be 1. Thus, a topological divisor of zero is a non-invertible element of $\mathcal{A}$ and it is also a topological divisor (hence a non-invertible element) of any normed algebra containing $\mathcal{A}$.

Proposition 5.4: If $\mathcal{M}$ is a maximal ideal of a Banach algebra $\mathcal{A}$, then $\mathcal{M}$ is closed in $\mathcal{A}$.

Proof: It suffices to show that $\overline{M}$ is not A. If that was true, then there would be a sequence (x_m) in M, such that $x_n \to 1$. By the aforementioned reformulation of Proposition 5.1, all elements x_n, for which $||x_n - 1|| < 1$, are invertible, which is absurd, since $\mathcal{M}$ is a proper ideal of $\mathcal{A}$.

If a is an element of an algebra $\mathcal{A}$ the *spectrum* of a is the set

$$sp(a) = \{\lambda |\ \lambda \in \mathbb{K},\ a - \lambda 1 \textit{ is not invertible}\}.$$

Spectrum depends on algebra $\mathcal{A}$, so if necessary $sp_{\mathcal{A}}(a)$ may be used. If $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, then, obviously,

$$sp_{\mathcal{A}}(a) \subseteq sp_{\mathcal{B}}(a),$$

where $a \in B$.

Example 1: Consider the algebra $\mathfrak{B}(X)$, where $dim(X) < \infty$. If $f \in \mathfrak{B}(X)$, then $\lambda \in sp(f)$ iff $f - \lambda 1$ is not invertible iff $f - \lambda 1$ is not $1-1$ iff $Ker(f - \lambda 1) \neq \{0\}$ iff $\exists x \neq 0 : f(x) = \lambda x$. I.e., the spectrum of f is the set of eigenvalues of f, which is exactly the set of roots of the characteristic polynomial of f. Therefore, $sp(f)$ is finite. If $\mathbb{K} = \mathbb{C}$, then the fundamental theorem of algebra implies that $sp(f) \neq \emptyset$. If $\mathbb{K} = \mathbb{R}$, then $sp(f)$ may be empty. E.g., matrix
in $\mathbb{R}^{2\times 2}$ has $x^2 + 1$ as characteristic polynomial, therefore it has no eigenvalues.

Example 2: If $dim(X) < \infty$ and $f \in \mathfrak{B}(X)$, then f may be $1-1$ but not onto X, i.e., there may exist spectrum values which are not eigenvalues. E.g., the shift operator $S \in \mathfrak{B}(l^2)$, where

$$S(x_1, x_2, ..., x_n, ...) = (0, x_1, x_2, ..., x_n, ...),$$

is an isometry ($||S(x)|| = ||x||$), therefore it is $1-1$, but it is not onto l^2, since $(1, 0, 0, ..., 0, ...) \notin S(l^2)$. Moreover, it is easy to see that S has not eigenvalues at all, while[17]

$$\lambda \in sp(S) \Leftrightarrow |\lambda| \leq 1.$$

Thus, the spectrum of an operator may be infinite, while, even in the complex case, its set of eigenvalues is empty.

Example 3: If $f \in C_b(X)$, then $\lambda \notin sp(f) \Leftrightarrow \frac{1}{f-\lambda 1}$ is defined everywhere and it is bounded $\Leftrightarrow \exists M > 0\ \forall x \in X\ \frac{1}{|f(x)-\lambda|} \leq M \Leftrightarrow \exists \varepsilon > 0\ \forall x \in X\ |f(x) - | \geq \varepsilon \Leftrightarrow \lambda \notin \overline{f(X)}$. Thus,

$$sp(f) = \overline{f(X)}.$$

[17]These facts hold in both real and complex l^2.

If X is compact, then
$$sp(f) = f(X).$$
Hence, if $f \in H(D)$, then $sp(f) = f(D)$, and if $f \in C[a,b]$, then $sp(f) = f([a,b])$.

Example 4: This example shows the dependence of the spectrum on the algebra. From the maximum principle, if f is in $H(D)$, then

$$||f||_\infty = max\{|f(z)| \mid |z| = 1\}.$$

We then define,
$$T : H(D) \to C(\partial D), \quad f \mapsto f|_{\partial D}.$$
T is an isometric embedding, therefore $H(D)$ is identical to a closed subalgebra of $C(\partial D)$. Considering the identity function on D, $f(z) = z$, then $f \in H(D)$, and

$$sp_{H(D)}(f) = f(D) = D$$

while,
$$sp_{C(\partial D)}(f) = f(\partial D) = \partial D.$$
The term spectrum derives from Physics. In quantum mechanics observables are represented by self-adjoint operators on a Hilbert space. The spectrum of an operator is interpreted as the set of values which result from the measurement of a magnitude, such as energy. The measurement of energy, because of formulas like

$$E = h\nu,$$

is reduced to measurements of frequency of some transmitted radiation. The view we get from the study of radiations is called spectrum and shows in which frequencies we get radiation.

Proposition 5.5: If $\mathcal{A}$ is a Banach algebra and $a \in A$, then $sp(a)$ is a compact subset of $\mathbb{K}$.

Proof: Since
$$sp(a) = \{\lambda \mid \lambda \in \mathbb{K},\ a - \lambda 1 \in A - U(A)\},$$
$sp(a)$ is the inverse image of a closed set under the continuous function $\lambda \mapsto a - \lambda 1$, therefore it is closed. It is also bounded, since

$$\lambda \in sp(a) \Rightarrow |\lambda| \leq ||a||.$$

If $|\lambda| > ||a||$, then $||\lambda^{-1}a|| < 1$, therefore $1 - \lambda^{-1}a$ is invertible, i.e., $a - \lambda 1$ is invertible, which is absurd.

In non-complete algebras all the above results fail. As we have already mentioned, $(P[0,1], ||.||_\infty)$ is a non-complete normed algebra. Since

$$U(P[0,1]) = \{\lambda 1 \mid \lambda \in \mathbb{K} - \{0\}\},$$

$U(P[0,1])$ has an empty interior, hence it is not open. If $p(t) = \frac{t}{2}$, then $||p||_\infty = \frac{1}{2} < 1$, but $1 - p$ is not invertible. If $degp \geq 1$, then $sp(p) = \mathbb{C}$, which is unbounded.

Next proposition refers exclusively to complex normed algebras.

Proposition 5.6: If $\mathcal{A}$ is a complex normed algebra and $a \in A$, then

$$sp(a) \neq \emptyset.$$

Proof: We shall need the following generalization of Liouville's theorem:

Lemma: If X is a complex normed space and $f : \mathbb{C} \to X$ holomorphic and bounded, then f is constant.

Proof of Lemma: Suppose that f is not constant. I.e., there are $\lambda, \mu \in \mathbb{C}$ such that $f(\lambda) \neq f(\mu)$. By Hahn-Banach theorem there is $\varphi^* \in X^*$: $\varphi(f(\lambda)) \neq \varphi(f(\mu))$. Obviously, $\varphi \circ f : \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded, therefore, by Liouville's theorem, constant, which contradicts our hypothesis.

Going back to the proof of Proposition 5.6, suppose that $sp(a) = \emptyset$. We then define $f : \mathbb{C} \to A$

$$f(\lambda) = (a - \lambda 1)^{-1}.$$

It suffices to show that f is holomorphic and bounded (because then, it must be constant, which is absurd, since f is $1-1$).
We show first that f is holomorphic.

$$\frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} = \frac{(a - \lambda 1)^{-1} - (a - \lambda_0 1)^{-1}}{\lambda - \lambda_0} =$$

$$(a - \lambda 1)^{-1}(a - \lambda_0 1)^{-1}\frac{(a - \lambda_0 1) - (a - \lambda 1)}{\lambda - \lambda_0} =$$

$$(a - \lambda 1)^{-1}(a - \lambda_0 1)^{-1}\frac{(\lambda_0 - \lambda)}{\lambda - \lambda_0} 1 =$$

$$(a - \lambda 1)^{-1}(a - \lambda_0 1)^{-1} \overset{\lambda \to \lambda_0}{\longrightarrow} (a - \lambda_0 1)^{-2}.$$

The above route is similar to the calculation of the derivative at λ_0 of $f : \mathbb{C} - \{0\} \to \mathbb{C}$, $f(\lambda) = \frac{1}{a - \lambda}$. Thus,

$$\lim_{\lambda \to \lambda_0} \frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} = (a - \lambda_0 1)^{-2},$$

using the continuity of function $^{-1}$. Hence, f is holomorphic.
Also, since

$$\lim_{\lambda \to \infty} f(\lambda) = \lim_{\lambda \to \infty} (a - \lambda 1)^{-1} =$$

$$\lim_{\lambda \to \infty} [\lambda(\lambda^{-1} a - 1)]^{-1} = \lim_{\lambda \to \infty} \lambda^{-1}(\lambda^{-1} a - 1)^{-1} = 0,$$

since $\lim_{\lambda \to \infty} \lambda^{-1} = 0$ and $\lim_{\lambda \to \infty} \lambda^{-1} a - 1 = 0$, therefore, $\lim_{\lambda \to \infty} (\lambda^{-1} a - 1)^{-1} = -1$. Thus, for each $\varepsilon > 0$ there is K compact subset of $\mathbb{C}$ such that for each $\lambda \in \mathbb{C} - K$ $|f(\lambda)| < \varepsilon$. I.e., f is bounded in $\mathbb{C} - K$. But since f is holomorphic, it is also continuous, therefore $f(K)$ is a bounded set. Hence, f is bounded.

There are elementary, though longer, proofs of the above result avoiding Liouville's theorem, the involvement of which seems mysterious. But this is not true. Proposition 5.6 is the generalization of the well known fact that each complex matrix has eigenvalues, which is proved through the fundamental theorem of algebra, the shortest proof of which is through Liouville's theorem!

Proposition 5.7 (Gelfand-Mazur): If $\mathcal{A}$ is a complex division normed algebra, then $\mathcal{A}$ is isometrically isomorphic to $\mathbb{C}$.

Proof: If $a \in A$, then $sp(a) \neq \emptyset$. I.e., there is $\lambda \in \mathbb{C}$ such that $a - \lambda 1 = 0$. Thus, $a = \lambda 1$. The mapping

$$\lambda \mapsto \lambda 1$$

is an isometric isomorphism from $\mathbb{C}$ into $\mathcal{A}$. Actually, it is the unique homomorphism of $\mathbb{C}$-algebras from $\mathbb{C}$ into $\mathcal{A}$. I.e., the identification between $\mathcal{A}$ and $\mathbb{C}$ is natural.

We also write down without proof the real analogue to the Gelfand-Mazur theorem.

Proposition 5.8: If $\mathcal{A}$ is a division real normed algebra, then $\mathcal{A}$ is algebraically isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, where $\mathbb{H}$ is the algebra of quaternions.

6 Spectral radius

If $\mathcal{A}$ is a complex Banach algebra and $a \in A$, then the spectral radius of a is

$$r(a) = sup\{|\lambda| \mid \lambda \in sp(a)\}.$$

Obviously, the concept of spectral radius is well defined by the non-emptiness of the spectrum and its compactness. Of course,

$$r(a) \leq ||a||,$$

as we have seen in the proof of Proposition 5.5.
Our aim is to prove a formula about spectral radius (Proposition 6.1). Although there is an elementary proof of that, the use of the theory of holomorphic functions valued in a Banach space is more enlightening. This theory, at the extent needed here, is a simple generalization of classical complex analysis.
Its methods are basically two: either we repeat classical proofs replacing the absolute values with norms or we combine the classical result with Hahn-Banach theorem generalizing it in Banach spaces. We have already encountered an example of the second case in the lemma of Proposition 5.6. The reason that Hahn-Banach theorem plays such a role is that, if $G \subseteq \mathbb{C}$, open, X is a complex Banach space and $f : G \to \mathbb{C}$, holomorphic, then for each $\varphi \in X^*$, $\varphi \circ f : G \to \mathbb{C}$ is holomorphic.
Also, if $\gamma : [a, b] \to \mathbb{C}$ a C^1 closed curve and $f : \gamma([a, b]) \to X$ continuous, then for each $\varphi \in X^*$, $\varphi \circ f : \gamma([a, b]) \to \mathbb{C}$ is continuous and

$$\int_\gamma (\varphi \circ f)(z)dz = \varphi(\int_\gamma f(z)dz).$$

Obviously,

$$\int_\gamma f(z)dz$$

is the limit of the following Riemann sums

$$\sum_{i=1}^{n} f(\gamma(\xi_i))(\gamma(t_i) - \gamma(t_{i-1})),$$

where $\{a = t_0 < t_1 < ... < t_n = b\}$ is a partition of $[a, b]$ and ξ_i a selection of intermediate points. The limit is taken as $max\{t_i - t_{i-1} \mid i = 1, 2, ..., n\} \to 0$.
Another example of the above method is the following:
If $G \subseteq \mathbb{C}$, open, X is a complex Banach space and $f : G \to \mathbb{C}$ is holomorphic, and $\gamma : [a, b] \to \mathbb{C}$ a curve on which classical Cauchy theorem is applicable, then for each $\varphi \in X^*$,

$$\varphi(\int_\gamma f(z)dz) = \int_\gamma (\varphi \circ f)(z)dz = 0,$$

therefore

$$\int_\gamma f(z)dz = 0.$$

Using the above methods we can show the following results:

(I) If X is a complex Banach space, (a_n) a sequence of elements of X and $z_0 \in \mathbb{C}$, then the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

is

$$R = (limsup||a_n||^{\frac{1}{n}})^{-1},$$

where $0, \infty$ are inverse of each other. Function $f : B(z_0, R) \to X$ with

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

has derivative of every order, and for each n, $a_n = \frac{f^{(n)}(z_0)}{n!}$. This implies that f is uniquely represented as a power series.

(II) If X is a complex Banach space, $G \subseteq \mathbb{C}$ open, and $f : G \to \mathbb{C}$ holomorphic, then f is representable as a power series in G i.e., for each $z_0 \in G$, for each $r > 0$ such that $B(z_0, r) \subseteq G$, there is a sequence (a_n) in X, such that

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n,$$

for each $z \in B(z_0, r)$. Hence, applying (I) on f, f has derivative of each order and the coefficients a_n are uniquely determined.

Proposition 6.1: If $\mathcal{A}$ is a complex Banach algebra and $a \in A$, then

$$lim_{n\to\infty}||a^n||^{\frac{1}{n}}$$

exists, and

$$r(a) = lim_{n\to\infty}||a^n||^{\frac{1}{n}} = inf_{n\geq 1}||a^n||^{\frac{1}{n}}.$$

Proof: We need the following lemma:

Lemma: If $\mathcal{A}$ is an algebra, $a \in A$, $\lambda \in sp(a)$ and $n \in \mathbb{N}$, then $\lambda^n \in sp(a^n)$.

Proof: Since

$$\begin{aligned} a^n - \lambda^n 1 &= (a - \lambda 1)(a^{n-1} + \lambda a^{n-2} + ... + \lambda^{n-1}) \\ &= (a^{n-1} + \lambda a^{n-2} + ... + \lambda^{n-1})(a - \lambda 1), \end{aligned}$$

then, if $a^n - \lambda^n 1$ was invertible, $a - \lambda 1$ would be invertible too, which is absurd, since $\lambda \in sp(A)$.

Continuing the proof of Proposition 6.1, if $\lambda \in sp(a)$, then $\lambda^n \in sp(a^n)$, hence, $|\lambda^n| \leq ||a^n||$ i.e., $|\lambda| \leq ||a^n||^{\frac{1}{n}}$. Therefore, $r(a) \leq inf_{n\geq 1}||a^n||^{\frac{1}{n}} \leq liminf||a^n||^{\frac{1}{n}}$. So, it suffices to show that

$$limsup||a^n||^{\frac{1}{n}} \leq r(a).$$

Then, $lim_n||a^n||^{\frac{1}{n}}$ exists, since $liminf||a^n||^{\frac{1}{n}} = limsup||a^n||^{\frac{1}{n}}$ and actually $lim_n||a^n||^{\frac{1}{n}} = r(a)$.

By result (I) though, $limsup||a^n||^{\frac{1}{n}} = \frac{1}{R}$, where R is the radius of convergence of power series

$$\sum_{n=0}^{\infty} a^n z^n.$$

Thus, it suffices to show that $\frac{1}{R} \leq r(a)$. But, if $||za|| < 1$,

$$\sum_{n=0}^{\infty} a^n z^n = \sum_{n=0}^{\infty} (za)^n = (1 - za)^{-1}.$$

Function

$$z \mapsto (1 - za)^{-1}$$

is defined on the open set $G = \{z|\ z \in \mathbb{C} : 1 - za \in U(A)\}$ and it is holomorphic (this can be proved in the same way we proved that $\lambda \mapsto (a - \lambda 1)^{-1}$ was holomorphic in Proposition 5.6). So, by result (II) it is representable by power series in $B(0, r)$, for each $r > 0$ such that $B(0, r) \subseteq G$. This power series is $\sum_{n=0}^{\infty} a^n z^n$, due to uniqueness mentioned in result (I). I.e., if $|z| < r$, then $\sum_{n=0}^{\infty} a^n z^n$ converges. Hence, $B(0, r) \subseteq G$ implies that $r \leq R$. Consequently, if $R < r$, then $B(0, r) \subsetneq G$, i.e., there is z with $|z| < r$ and $z \notin G$. Then, $z \neq 0$ and if $\lambda = \frac{1}{z}$, then $\lambda \in sp(a)$, since

$$z \notin G \Leftrightarrow (1 - za)^{-1} \notin U(A) \Rightarrow za - 1 \notin U(A) \Rightarrow a - \frac{1}{z} 1 \notin U(A).$$

Thus, $|\lambda| \leq r(a)$, and since by hypothesis $|z| < r$, $\frac{1}{r} < \frac{1}{|z|} = |\lambda|$ i.e., $\frac{1}{r} < r(a)$.
So, we have shown that for each $r > R$, $\frac{1}{r} < r(a)$. Taking the limit $r_n \to R$, where $r_n > R$, we reach

$$\frac{1}{R} = lim_n \frac{1}{r_n} \leq r(a). \qquad \diamond$$

By its definition the spectral radius of a is the radius of the smallest closed circular disc in $\mathbb{C}$, with 0 as its center, which contains $sp(a)$. By Proposition 6.1 we see that $r(a)$ is computed independently from $sp(a)$.
Another important feature of the formula

$$r(a) = lim_{n\to\infty} ||a^n||^{\frac{1}{n}}$$

is that it connects the algebraic structure of A, within which $sp(a)$ and $r(a)$ are defined, with the metric structure of $\mathcal{A}$, since its right hand part depends on the norm structure of $\mathcal{A}$. Obviously, the quantity

$$lim_{n\to\infty} ||a^n||^{\frac{1}{n}}$$

is the same for all norms which define a Banach algebra-structure on $\mathcal{A}$.
Also, the $r(a)$ formula shows that spectral radius, contrary to spectrum, does not depend on the algebra in which a is contained. I.e., if $\mathcal{A}$ is a complex Banach algebra and $\mathcal{B}$ a closed subalgebra of $\mathcal{A}$ such that $a \in B$, then

$$r_A(a) = r_B(a).$$

It is also possible that $a \neq 0$ and $r(a) = 0$. Equivalently, $a \neq 0$ and $sp(a) = \{0\}$. E.g.,

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$$

and $sp(a) = \{0\}$. If $r(a) = 0$, then a is called *generalized nilpotent*. Obviously, if a is nilpotent i.e., there is some n such that $a^n = 0$, then $r(a) = 0$.

7 Hermitian and Symmetric algebras

First, we prove some properties of the spectrum.

Proposition 7.1: If a, b belong to an algebra $\mathcal{A}$, then

$$sp(ab) \cup \{0\} = sp(ba) \cup \{0\}.$$

Proof: It suffices to show that $sp(ba) - \{0\} \subseteq sp(ab) - \{0\}$ i.e., if $ab - \lambda 1$ is invertible, then $ba - \lambda 1$ is invertible, or equivalently, if $ab - 1$ is invertible, then $ba - 1$ is invertible, or equivalently, if $1 - ab$ is invertible, then $1 - ba$ is invertible.
If $||ab|| < 1$ and $||ba|| < 1$, then

$$(1 - ab)^{-1} = \Sigma_{n=0}^{\infty}(ab)^n = 1 + ab + abab + ababab + ...$$

$$(1 - ba)^{-1} = \Sigma_{n=0}^{\infty}(ba)^n = 1 + ba + baba + bababa + ...$$

$$= 1 + b(1a) + b(ab)a + b(abab)a + ...$$

$$= 1 + b(1 + ab + abab + ...)a$$

$$= 1 + b(1 - ab)^{-1}a.$$

It is really trivial to check that if $1 - ab$ is invertible, then $1 + b(1 - ab)^{-1}a$ is the inverse of $1 - ba$.$\diamond$

It is natural to ask for examples of algebras for which the equivalence

$$0 \in sp(ab) \Leftrightarrow 0 \in sp(ba)$$

holds and of algebras for which doesn't.
If $A, B \in \mathbb{K}^{n\times n}$, then $0 \notin sp(AB) \Leftrightarrow A$ is right-invertible and B is left-invertible $\Leftrightarrow$ A, B are invertible $\Leftrightarrow$ BA is invertible $\Leftrightarrow 0 \notin sp(BA)$.
On the other hand, if S is the shift operator in l^2,

$$S(x_1, x_2, ..., x_n, ...) = (0, x_1, x_2, ..., x_n, ...)$$

and S^* its conjugate,

$$S^*(x_1, x_2, ..., x_n, ...) = (x_2, x_3, ..., x_n, ...),$$

then

$$SS^*(x) = (0, x_2, x_3, ..., x_n, ...)$$
$$S^*S(x) = (x_1, x_2, ..., x_n, ...) = I(x).$$

Hence, since S^*S is invertible, $0 \notin sp(S^*S)$, while $0 \in sp(SS^*)$, since SS^* is not invertible.

Proposition 7.2: If $\mathcal{A}$ is an algebra and $a \in A$, then

$$sp(a + \lambda 1) = sp(a) + \lambda.$$

Proof: Obviously, if $\mu \in sp(a)$, then $\mu + \lambda \in sp(a + \lambda 1)$ and if $\mu \in sp(a + \lambda 1)$, then $\mu - \lambda \in sp(a)$.$\diamond$

Proposition 7.3: If $\mathcal{A}$ is a complex Banach algebra and $\lambda \neq 0$, then there are no $a, b \in A$ such that

$$ab - ba = \lambda 1.$$

Proof: If there were such a and b, then, by Proposition 7.2, $sp(ab) = sp(ba) + \lambda$, which is absurd by Proposition 7.1 and the compactness and the non-emptiness of the spectrum.$\diamond$

Proposition 7.3 derives from physics. In Quantum Mechanics observables position and momentum are described as operators P and Q, respectively, on a complex Hilbert space, satisfying Heisenberg's relation

$$QP - PQ = i\hbar I,$$

where $\hbar = \frac{h}{2\pi}$ and h is Planck's constant. Proposition 7.3 shows that P and Q are not elements of a Banach algebra, therefore, they cannot be both bounded.

Proposition 7.4: If $\mathcal{A}$ is Banach algebra, $\mathcal{B}$ a closed subalgebra of $\mathcal{A}$ and $a \in B$, using the symbols

$$\alpha = sp_{\mathcal{A}}(a) \qquad \beta = sp_{\mathcal{B}}(a),$$

then

$$\partial\beta \subseteq \partial\alpha \subseteq \alpha \subseteq \beta.$$

Proof: We only need to show that $\partial\beta \subseteq \partial\alpha$. If $\lambda \in \partial\beta$, then $\lambda \in \beta$ and there exists a sequence (λ_n) in $\mathbb{C} - \beta$ such that $\lambda_n \to \lambda$. But if $(\lambda_n) \in \mathbb{C} - \beta$, then $(\lambda_n) \in \mathbb{C} - \alpha$. So, in order to show that $\lambda \in \partial\alpha$ it suffices to show that $\lambda \in \alpha$. Suppose that $\lambda \notin \alpha$ i.e., $a - \lambda 1 \in U(A)$. Since $\lambda_n \to \lambda$ and $^{-1}$ is continuous, then

$$(a - \lambda_n 1)^{-1} \to (a - \lambda 1)^{-1},$$

where $(a - \lambda_n 1)^{-1}$ is a sequence of elements of B and $(a - \lambda 1)^{-1} \in B$, since B is closed, which is absurd, since $\lambda \in \beta$.$\diamond$.

We see that the spectrum of an element of an algebra Banach $\mathcal{B}$ is shortened when we compute it with respect to a larger Banach algebra $\mathcal{A}$ containing $\mathcal{B}$. Its boundary though, does not vanish, only it is shortened, it is as if a hole is being created. As we have seen in Example 4 of Section 5,

$$H(D) \preceq C(\partial D)$$

and if $a = id$, then $\alpha = \partial D$, $\beta = D$, while $\partial\alpha = \partial D$ and $\partial\beta = \partial D$.
By Proposition 7.4, if $int\beta = \emptyset$, then $\partial\beta = \partial\alpha = \alpha = \beta$. E.g., if $\beta \subseteq \mathbb{R}$, then $\alpha = \beta$, since $int\beta = \emptyset$ in $\mathbb{C}$ (of course here A and B are considered to be complex Banach algebras). Additionally, if β is a countable set, then $int\beta = \emptyset$ and again $\alpha = \beta$.

Proposition 7.5: If $\mathcal{A}$ is complex Banach algebra and $a \in A$, then for each closed subalgebra $\mathcal{B}$ of $\mathcal{A}$ with $a \in B$

$$sp_{\mathcal{A}}(a) = sp_{\mathcal{B}}(a) \Leftrightarrow \mathbb{C} - sp_{\mathcal{A}}(a) \text{ is connected.}$$

Proof: We shall only prove the ($\Leftarrow$) direction since the other one is more difficult and we wont use it here.
Suppose that $\mathcal{B}$ is a closed subalgebra of $\mathcal{A}$, $a \in B$ and $sp_{\mathcal{A}}(a) \subsetneq sp_{\mathcal{B}}(a)$. Since $\mathbb{C} - sp_{\mathcal{B}}(a)$ is open, it suffices to show that $sp_{\mathcal{B}}(a) - sp_{\mathcal{A}}(a)$ is open, reaching then an absurdity. Suppose $\lambda \in \beta - \alpha$. By Proposition 7.4, $\lambda \notin \partial\beta$. Hence,

$$\beta - \alpha = int\beta - \alpha = int\beta \cap (\mathbb{C} - \alpha),$$

which is, of course an open set.$\diamond$

We see that if $\mathcal{A} = C(\partial D)$ and $\mathcal{B} = H(D)$, then $sp_{\mathcal{A}}(id) = \partial D$. But $\mathbb{C} - sp_{\mathcal{A}}(id)$ is not connected, which is compatible with the known fact $sp_{\mathcal{B}}(id) \neq sp_{\mathcal{A}}(id)$.
Also, if $\alpha \subseteq \mathbb{R}$ or α is a finite set, then $\mathbb{C} - \alpha$ is arcwise connected, therefore connected, which gives again the left part of the above equivalence.

Proposition 7.6: If $\mathcal{A}$ is an algebra and $\mathcal{B}$ is a closed subalgebra of $\mathcal{A}$, then

$$U(B) = U(A) \cap B \Leftrightarrow \forall a \in B, \quad sp_{\mathcal{A}}(a) = sp_{\mathcal{B}}(a).$$

Proof: ($\Rightarrow$) If $\lambda \in sp_{\mathcal{B}}(a) - sp_{\mathcal{A}}(a)$, then $(a - \lambda 1) \notin U(B)$ which is equivalent to $(a - \lambda 1) \in U(B)'$, where $U(B)' = U(A)' \cup B'$, therefore $(a - \lambda 1) \in U(A)'$, which is absurd.
($\Leftarrow$) Generally, $U(B) \subseteq U(A) \cap B$. Suppose $b \in (U(A) \cap B) - U(B)$. Then $b = (b + \lambda 1) - \lambda 1 \in U(A)$, hence $\lambda \notin sp_{\mathcal{A}}(b + \lambda 1)$. Also, $(b + \lambda 1) - \lambda 1 \notin U(B)$, therefore $\lambda \in sp_{\mathcal{B}}(b + \lambda 1)$, which is absurd.$\diamond$

Proposition 7.7: If $\mathcal{A}$ is a $*$-algebra and $\mathcal{B}$ is a $*$-subalgebra of $\mathcal{A}$, the following are equivalent:

(i) If $a \in B$ and a is invertible in $\mathcal{A}$, then a is also invertible in $\mathcal{B}$.
(ii) If $a \in B$ and $\lambda \in sp_{\mathcal{B}}(a)$, then $\lambda \in sp_{\mathcal{A}}(a)$.
(iii) If $a \in B$ and $a^* = a$, where a is invertible in $\mathcal{A}$, then a is invertible in $\mathcal{B}$.
(iv) If $a \in B$, $a^* = a$ and $\lambda \in sp_{\mathcal{B}}(a)$, then $\lambda \in sp_{\mathcal{A}}(a)$.

Proof: By Proposition 7.6 it suffices to show the implication (iii) $\Rightarrow$ (i). Suppose $a \in B$. Then by the hypothesis of (i)a^{-1} is in A. a^*a is self-adjoint and belongs to B. Also, $(a^*)^{-1}$ is in A, since $(a^*)^{-1} = (a^{-1})^*$. Thus, $(a^*a)^{-1}$ is in A and therefore $(a^*a)^{-1}$ is in B. Moreover $(a^*a)^{-1} = a^{-1}(a^{-1})^*$. Since $a \in B$, $a(a^{-1}a^{-1})^* \in B$, therefore $(a^{-1})^* \in B$ and consequently $a^{-1} \in B$.$\diamond$

If $\mathcal{A}$ is a complex $*$-algebra, then $\mathcal{A}$ is called *Hermitian* iff for each $a \in A$ such that $a^* = a$,

$$sp(a) \subseteq \mathbb{R}.$$

The following examples motivate the above definition:
If $\mathcal{A} = \mathbb{C}$, then $a^* = a \Leftrightarrow sp(a) \subseteq \mathbb{R} \Leftrightarrow a \in \mathbb{R}$.
If $\mathcal{A} = C_b(X)$, then $f^* = f \Leftrightarrow f(X) \subseteq \mathbb{R} \Leftrightarrow \overline{f(X)} \subseteq \mathbb{R} \Leftrightarrow sp(f) \subseteq \mathbb{R}$.
If $\mathcal{A} = \mathfrak{B}(X)$, then if $T^* = T$, then it is easy to see that the set of eigenvalues of T is a subset of $\mathbb{R}$ and less easily that the spectrum of T is a subset of $\mathbb{R}$. More generally, we have the following result.

Proposition 7.8: Every complex C^*-algebra is Hermitian.

Proof: Suppose $\mathcal{A}$ is a complex C^*-algebra, $a \in A$, hermitian, and $\lambda + \mu i \in sp(a)$, where $\mu \neq 0$. Then, $\lambda + (\mu + n)i \in sp(a + ni1)$ for each n in $\mathbb{Z}$. Hence, by Proposition 7.5,

$$|\lambda + (\mu + n)i|^2 \leq ||a + ni1||^2 = ||(a + ni1)^*(a + ni1)||$$
$$= ||a^2 + n^2 1|| \leq ||a||^2 + n^2.$$

I.e., $\lambda^2 + \mu^2 + n^2 + 2\mu n \leq ||a||^2 + n^2$, i.e., $\lambda^2 + \mu^2 + 2\mu n \leq ||a||^2$, which is absurd, since $\mu \neq 0$.$\diamond$

If we consider the algebra $H(D)$, then if $f(z) = z$, $f^* = f$ and $sp(f) = D \subsetneq \mathbb{R}$. This does not contradict the last result, since $H(D)$ is not a C^*-algebra.

Proposition 7.9: If $\mathcal{A}$ is a complex $*$-algebra, the following are equivalent:
(i) $\mathcal{A}$ is Hermitian.
(ii) For each a in A such that $a^* = a$, $i \notin sp(a)$.
(iii) For each a in A such that $a^* = a$, $a^2 + 1$ is not invertible.

Proof: (i) $\Rightarrow$ (ii) is trivial.
(ii) $\Rightarrow$ (iii): If $i \notin sp(a)$, then $\bar{i} = -i \notin sp(a)$ (we use here the obvious equality $sp(a^*) = \overline{sp(a)}$). $a^2 + 1$ can be written as $a^2 + 1 = (a - i1)(a + i1)$. Since $a - i1, a + i1 \in U(A)$, $a^2 + 1 \in U(A)$ too.
(iii) $\Rightarrow$ (ii): Since $(a - i1), (a + i1)$ commute and $a^2 + 1 \in U(A)$, then $a - i1, a + i1 \in U(A)$ too.
(ii) $\Rightarrow$ (i): If $\lambda \in \mathbb{R} - \{0\}$, since $\lambda^{-1} \in Her(A)$, $\lambda^{-1} - i1 \in U(A)$, therefore, $\lambda\lambda^{-1} - i1 = a - \lambda i1 \in U(A) \Leftrightarrow \lambda i \notin sp(a)$. If $\mu + \lambda i \in sp(a)$, then $\lambda i \in sp(a - \mu 1)$, which is absurd, since $a - \mu 1 \in Her(A)$.$\diamond$

Proposition 7.10: If $\mathcal{A}$ is a complex Banach $*$-algebra and $\mathcal{B}$ a closed $*$-subalgebra of $\mathcal{A}$, then, if $\mathcal{A}$ is Hermitian, then $\mathcal{B}$ is Hermitian too, and for each a in B:

$$sp_{\mathcal{A}}(a) = sp_{\mathcal{B}}(a).$$

Proof: Just use our second remark following Proposition 7.5 for the implication in question, while Proposition 7.7 suffices for the equality of the spectrums.$\diamond$

Proposition 7.11: If $\mathcal{A}$ is a complex Banach C^*-algebra and $\mathcal{B}$ a closed $*$-subalgebra of $\mathcal{A}$, then $\mathcal{B}$, which is of course a C^*-algebra too, satisfies

$$sp_{\mathcal{A}}(a) = sp_{\mathcal{B}}(a),$$

for each a in B.

Proof: Proposition 7.10 is enough, since every complex C^*-algebra is Hermitian.$\diamond$

If $\mathcal{A}$ is a complex $*$-algebra, an element a of A is called *positive* iff there exists b in A such that

$$a = b^*b.$$

This only one among the definitions of element positivity that can be found in the literature.
If $\mathcal{A}$ is a complex $*$-algebra, $\mathcal{A}$ is called *symmetric* iff

$$sp(a) \subseteq [0, +\infty),$$

for each positive element of A.
The following examples motivate the above definition:
If $\mathcal{A} = \mathbb{C}$, then a is positive $\Leftrightarrow a \geq 0 \Leftrightarrow sp(a) \subseteq [0, +\infty)$.
If $\mathcal{A} = C_b(X)$, then f is positive $\Leftrightarrow f(X) \subseteq [0, +\infty) \Leftrightarrow \overline{f(X)} \subseteq [0, +\infty) \Leftrightarrow sp(f) \subseteq [0, +\infty)$.
If $\mathcal{A} = \mathfrak{B}(X)$, then it can be proved (less easily) that T is positive iff $< Tx|x > \geq 0$, for each $x \in H$ iff T is normal and $sp(T) \subseteq [0, +\infty)$.

Proposition 7.12: If $\mathcal{A}$ is a complex $*$-algebra the following are equivalent:

(i) $\mathcal{A}$ is symmetric.
(ii) For each a in A, $-1 \in sp(a^*a)$.
(iii) For each a in A, $1 + a^*a \in U(A)$.

Proof: (i) $\Rightarrow$ (ii) and (ii) $\Leftrightarrow$ (iii) are trivial.
(ii) $\Rightarrow$ (i): From (iii) of Proposition 7.9 we get that $sp(a^*a) \subseteq \mathbb{R}$. Therefore, it suffices to show that for each $\lambda > 0$, $-\lambda \notin sp(a^*a)$. Since $\lambda^{-1}a^*a$ is positive $\lambda^{-1}a^*a + 1 \in U(A)$, hence $\lambda(\lambda^{-1}a^*a + 1) = a^*a + \lambda 1 \in U(A) \Leftrightarrow -\lambda \notin sp(a^*a)$.$\diamond$

It is obvious that in previous proof we of the fact that Proposition 7.12 (iii) gives Proposition 7.9 (iii). I.e., the following is proved:

Proposition 7.13: If $\mathcal{A}$ is a complex $*$-algebra, then if $\mathcal{A}$ is symmetric, it is also Hermitian.

Proposition 7.14: If $\mathcal{A}$ is a complex Banach $*$-algebra and $\mathcal{B}$ is a closed $*$-subalgebra of $\mathcal{A}$, then, if $\mathcal{A}$ is symmetric, $\mathcal{B}$ is also symmetric and

$$sp_{\mathcal{A}}(a) = sp_{\mathcal{B}}(a),$$

for each a in B.

Proof: From the second remark following Proposition 7.5, $sp_A(a^*a) \subseteq \mathbb{R}^+$, hence $sp_B(a^*a) = sp_A(a^*a) \subseteq [0, +\infty)$.
Equality is proved by Proposition 7.10,since a symmetric algebra is also Hermitian.

Obviously, $H(D)$ is not symmetric algebra, since it is not Hermitian.

In 1943 Gelfand and Naimark proved the non-commutative representation theorem for symmetric C^*-algebras. They conjectured though, that the symmetric property must be proved from the rest axioms of a C^*-algebra. Actually, Fukamiya (1952), Kelly - Vaught (1953), Kaplansky (1953) proved that:

a C^*-algebra is symmetric.

Also, in 1970 Shirali - Ford proved that, if $\mathcal{A}$ is a complex Banach $*$-algebra, then if $\mathcal{A}$ is Hermitian, it is also symmetric.
Later (Proposition 9.3), we shall show that if $\mathcal{A}$ is a complex commutative Banach $*$-algebra, then if $\mathcal{A}$ is Hermitian, it is also symmetric.
In 1973 Wichmann found a complex Hermitian $*$-algebra (obviously not Banach) which is not symmetric.

8 Ideals

As we have already said at the beginning, a subset J of an algebra $\mathcal{A}$ is called an *ideal* of $\mathcal{A}$ iff it is an ideal of the corresponding ring. Then,

$$j \in J, \quad \lambda \in K \Rightarrow \lambda j = (\lambda 1)j \in J$$

i.e., J is a subalgebra and the quotient $\mathcal{A}/J$ is an algebra.
Also, if X is a normed space and Y a close subspace of X, then the quotient X/Y becomes a normed space with norm

$$||[x]|| = inf\{||x+y||, \ y \in Y\},$$

where $[x]$ is the equivalence class of x. If X is a Banach space, X/Y is also a Banach space.
If $\mathcal{A}$ is a normed algebra and J a closed ideal of $\mathcal{A}$, the quotient $\mathcal{A}/J$ is a normed space and also an algebra. Moreover, if J is proper, then $\mathcal{A}/J$ is also a normed algebra, since

$$||[x]|| \ ||[y]|| = inf\{||x+z||, \ z \in J\} inf\{||y+w||, \ w \in J\} =$$

$$inf\{||x+z|| \ ||y+w||, \ z, w \in J\} \geq$$
$$inf\{||(x+z)(y+w)||, \ z, w \in J\} =$$
$$inf\{||xy+zy+xw+zw||, \ z, w \in J\} \geq$$
$$inf\{||xy+z||, \ z \in J\} =$$
$$||[x][y]||.$$

Also,

$$||[1]|| = inf\{||1+z||, \ z \in J\} \leq ||1+0|| = 1$$

and

$$||[1]|| = ||[1]^2|| \leq ||[1]||^2 \Rightarrow 1 \leq ||[1]||.$$

Note that J was considered proper because if it wasn't $\mathcal{A}/J$ would be $\{0\}$, while we defined a normed algebra to include always a unit.
If J is an ideal of $\mathcal{A}$, then its closure is also an ideal and if $\mathcal{A}$ is a Banach algebra and J is proper, then $\overline{J}$ is also proper, since

$$J \subseteq A - U(A) \Rightarrow \overline{J} \subseteq A - U(A).$$

In that we way we get another proof of the fact that a maximal ideal in a Banach algebra is closed (Proposition 5.4).
If $\mathcal{A}$ is a commutative Banach algebra and J a maximal ideal of $\mathcal{A}$, then the quotient $\mathcal{A}/J$ is a Banach algebra and also a field. Recall that if $\mathcal{A}$ is a commutative ring with a unit, then an ideal J of $\mathcal{A}$ is maximal iff $\mathcal{A}/J$ is a field. Hence, if $\mathcal{A}$ is a complex algebra, then $\mathcal{A}/J$ is isomorphic to $\mathbb{C}$, by Gelfand-Mazur theorem (Proposition 5.7).
If $\mathcal{A}$ is a $\mathbb{C}$-algebra, then an ideal J of $\mathcal{A}$ is called a $\mathbb{C}$-*ideal* iff $\mathcal{A}/J$ is isomorphic (as a $\mathbb{C}$-algebra) to $\mathbb{C}$. Obviously, then there is a unique isomorphism of $\mathbb{C}$-algebras between $\mathcal{A}/J$ and $\mathbb{C}$. Recall that if $\mathcal{A}$ is a $\mathbb{C}$-algebra, there is a unique homomorphism of $\mathbb{C}$-algebras from $\mathbb{C}$ to $\mathcal{A}$

$$\lambda \mapsto \lambda 1,$$

and this homomorphism is unique iff it is onto A.
A $\mathbb{C}$-ideal is maximal, since if $\mathcal{A}$ is a ring with a unit and J a proper ideal of $\mathcal{A}$, then J is maximal iff $\mathcal{A}/J$ is a simple ring i.e., it has one only proper ideal, and, of course, a field is a simple ring.

A *character* of $\mathcal{A}$ is a homomorphism of $\mathbb{C}$-algebras

$$\varphi : A \to \mathbb{C}.$$

Since $\varphi(A)$ is a $\mathbb{C}$-algebra, φ is an epimorphism (the only, non-trivial subalgebra of $\mathbb{C}$ is $\mathbb{C}$). Hence its kernel, $J = Ker\varphi$, is a $\mathbb{C}$-ideal ($A/Ker\varphi \cong Im\varphi = \mathbb{C}$).
If $\Phi(\mathcal{A})$ denotes the set of all characters of $\mathcal{A}$ and $M_{\mathbb{C}}(\mathcal{A})$ the set of all $\mathbb{C}$-ideals of $\mathcal{A}$, the mapping Ker

$$Ker : \Phi(\mathcal{A}) \to M_{\mathbb{C}}(\mathcal{A})$$

$$Ker(\varphi) = ker\varphi,$$

is $1-1$ and onto $M_{\mathbb{C}}(\mathcal{A})$.

$(1-1)$: Suppose $Ker\varphi = Ker\varphi^{'}$. The mappings $\overline{\varphi} : A/ker\varphi \to \mathbb{C}$ and $\overline{\varphi}^{'} : A/ker\varphi^{'} \to \mathbb{C}$, where

$$\overline{\varphi}(a + ker\varphi) = \varphi(\alpha), \quad \overline{\varphi^{'}}(a + ker\varphi^{'}) = \varphi^{'}(\alpha)$$

are equal, since the kernels are equal and the homomorphism of $\mathbb{C}$-algebras from $A/ker\varphi$ to $\mathbb{C}$ is unique. Therefore, φ and $\varphi^{'}$ are also equal.
(onto $M_{\mathbb{C}}(\mathcal{A})$): suppose $J \in M_{\mathbb{C}}(\mathcal{A})$. Then there exists an isomorphism $\overline{\varphi} : A/J \to \mathbb{C}$, so, if we define $\varphi : A \to \mathbb{C}$

$$\varphi(a) = \overline{\varphi}(a + J),$$

we get a character of A with J as its kernel.

Therefore, we have established a natural identification between characters and $\mathbb{C}$-ideals of $\mathcal{A}$.
if $\mathcal{M}(\mathcal{A})$ denotes the set of all maximal ideals of $\mathcal{A}$, then

$$M_{\mathbb{C}}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A}).$$

The well-known theorem on the existence of maximal ideals guarantees that

$$\mathcal{M}(\mathcal{A}) \neq \emptyset.$$

As we have already seen, if $\mathcal{A}$ is commutative complex Banach algebra, then

$$M_{\mathbb{C}}(\mathcal{A}) = \mathcal{M}(\mathcal{A}).$$

It is possible though, that $M_{\mathbb{C}}(\mathcal{A})$ is empty, even if $\mathcal{A}$ is a complex Banach algebra (of course, non-commutative). E.g., if we consider $\mathcal{A} = \mathbb{C}^{2\times 2}$, then it suffices to show that $\{0\}$ is a maximal ideal of $\mathbb{C}^{2\times 2}$, while it is not a $\mathbb{C}$-ideal i.e., there is no character of $\mathcal{A}$.
Note that if φ is in $M_{\mathbb{C}}(\mathcal{A})$ and a in A, then

$$\varphi(a) \in sp(a),$$

since, if $\varphi(a) = \lambda$, then $\varphi(a - \lambda 1) = 0$, hence $a - \lambda 1$ is not invertible. So, if $\mathcal{A}$ is a Banach algebra, then

$$|\varphi(a)| \leq ||a||$$

i.e., φ is a continuous functional such that $||\varphi|| \leq 1$. Moreover, since $\varphi(1) = 1$,

$$||\varphi|| = 1.$$

Proposition 8.1: If $\mathcal{A}$ is a commutative complex Banach algebra and $a \in A$, the following are equivalent:

(i) a is not invertible.
(ii) The ideal generated by a is proper.
(iii) There exists a maximal ideal M, such that $a \in M$.
(iv) There exists a character φ of $\mathcal{A}$, such that $\varphi(a) = 0$.

Proof: All the steps of the proof are elementary. Note that the implication (ii) $\Rightarrow$ (iii) makes use of Zorn's lemma and the equivalence (iii) $\Leftrightarrow$ (iv) uses the mapping Ker.$\diamond$

Proposition 8.2: If $\mathcal{A}$ is a commutative complex Banach algebra and $a \in A$, then

$$sp(a) = \{\varphi(a) \mid \varphi \in \mathcal{M}(\mathcal{A})\}.$$

Proof: We have already shown that if $\varphi \in \mathcal{M}(\mathcal{A})$, then $\varphi(a) \in sp(a)$. If $\lambda \in sp(a)$, $a - \lambda 1$ is not invertible, therefore, by Proposition 8.1, there exists a character φ of $\mathcal{A}$, such that $\varphi(a - \lambda 1) = 0$ i.e., $\varphi(a) = \lambda$.$\diamond$

If $\mathcal{A}$ is a commutative complex Banach algebra, $a \in A$ and $\varphi \in \mathcal{M}(\mathcal{A})$, then $\varphi(a) \in sp(a)$. Hence,

$$\varphi \in \prod_{a \in A} sp(a)$$

I.e.,

$$\mathcal{M}(\mathcal{A}) \subseteq \prod_{a \in A} sp(a),$$

while $\prod_{a \in A} sp(a)$ is equipped with the product topology, the smallest topology which turns the projection mappings

$$\pi_a : \prod_{a \in A} sp(a) \to sp(a)$$

into continuous functions, and $\mathcal{M}(\mathcal{A})$ is equipped with the relative topology. Equivalently, the product topology of $\prod_{a \in A} sp(a)$ is the weak topology defined by all mappings π_a. Likewise, the topology of $\mathcal{M}(\mathcal{A})$ is the weak topology defined by the family of all projection mappings $\widehat{a}$, $a \in A$,

$$\widehat{a} : \mathcal{M}(\mathcal{A}) \to \mathbb{C},$$

where $\widehat{a}$ is the restriction of π_a on $\mathcal{M}(\mathcal{A})$. Since the product topology is the topology of pointwise convergence, if $(\varphi_i)_{i \in I}$ is a net in $\mathcal{M}(\mathcal{A})$ and $\varphi \in \mathcal{M}(\mathcal{A})$, then

$$\varphi_i \to \varphi \Leftrightarrow \varphi_i(a) \to \varphi(a),$$

for each a in A.

$sp(a)$ is a compact T_2 space, for each a in A, therefore, by Tychonoff's theorem, $\prod_{a \in A} sp(a)$ is a compact, and of course, T_2 space.

$\mathcal{M}(\mathcal{A})$ is a closed subspace of $\prod_{a\in A} sp(a)$, since, if $(\varphi_i)_{i\in I}$ is a net in $\mathcal{M}(\mathcal{A})$ and $\varphi_i \to \varphi$, then $\varphi \in \mathcal{M}(\mathcal{A})$, since e.g.,

$$\varphi(a+b) = lim_i \varphi_i(a+b) = lim_i(\varphi_i(a) + \varphi_i(b))$$
$$= lim_i \varphi_i(a) + lim_i \varphi_i(b) = \varphi(a) + \varphi(b).$$

Therefore, $\mathcal{M}(\mathcal{A})$ is a compact T_2 space.

For each φ in $\mathcal{M}(\mathcal{A})$, φ is a continuous mapping from A to $\mathbb{C}$, hence we may view $\mathcal{M}(\mathcal{A})$ as a subset of $\mathcal{A}^*$, the dual space of $\mathcal{A}$. In that way the topology just defined on $\mathcal{M}(\mathcal{A})$ is the weak star topology w^*. By this compactness of $\mathcal{M}(\mathcal{A})$ is closely connected with

Alaoglou's theorem: The closed unit ball of $\mathcal{A}^*$ is w^*-compact.

Indeed, the proof of the compactness of $\mathcal{M}(\mathcal{A})$ we gave is completely analogous to the proof of Alaoglou's theorem. Moreover the compactness of $\mathcal{M}(\mathcal{A})$ is derived from Alaoglou's theorem, if we view $\mathcal{M}(\mathcal{A})$ as a w^*-closed subset of the closed unit ball of $\mathcal{A}^*$.

We can define on $\mathcal{M}(\mathcal{A})$ another natural topology which we shall study first in a more general setting.

In subsequent Propositions 8.3-8.8 $\mathcal{A}$ is a commutative ring with a unit and if $\mathcal{M}(\mathcal{A})$ is its set of maximal ideals and $a \in A$, we define

$$F(a) = \{M|\ M \in \mathcal{M}(\mathcal{A}),\ a \in M\}.$$

$\mathcal{M}(\mathcal{A})$ becomes a topological space if we consider the family

$$(F(a))_{a\in A}$$

as a base for its closed sets. $(F(a))_{a\in A}$ can be considered first as a subbase for the closed sets of some topology. Since though,

$$F(a_1) \cup ... \cup F(a_n) = F(a, ..., a_n)$$

i.e., the finite unions of elements of the family belong to the family, $(F(a))_{a\in A}$ is also a base for the closed sets of a topology. $(F(a))_{a\in A}$ is called the *structure space* of $\mathcal{A}$ and the above topology is called Stone or Zariski topology.

Proposition 8.3: If $X \subseteq \mathcal{M}(\mathcal{A})$, then the closure of X within the Zariski topology is

$$\overline{X} = \{M|\ M \in \mathcal{M}(\mathcal{A}),\ M \supseteq \bigcap X\}.$$

Proof: Obviously,

$$\{M|\ M \in \mathcal{M}(\mathcal{A}),\ M \supseteq \bigcap X\} = \bigcap_{a\in X} F(a),$$

therefore $\{M|\ M \in \mathcal{M}(\mathcal{A}),\ M \supseteq \bigcap X\}$ is a closed set which contains X. We show now that it is the smallest closed set containing X. If there is a closed set F such that $X \subseteq F \subsetneq \bigcap_{a\in X} F(a)$, then F itself is written within Zariski topology as

$$F = \bigcap_{b\in B} F(b).$$

By hypothesis, there exists $b \in B$ such that $b \notin \bigcap_X$. But for each $M^{'}$ in X

$$M^{'} \in F(b) \Leftrightarrow b \in M^{'} \Leftrightarrow b \in \bigcap X,$$

which is absurd.$\diamond$

Proposition 8.4: If $X \subseteq \mathcal{M}(\mathcal{A})$, then X is dense in $\mathcal{M}(\mathcal{A})$ within the Zariski topology iff

$$\bigcap X = \bigcap \mathcal{M}(\mathcal{A}).$$

Proof: X is dense in $\mathcal{M}(\mathcal{A})$ iff $\overline{X} = \mathcal{M}(\mathcal{A})$ iff $\bigcap_{a \in X} F(a) = \mathcal{M}(\mathcal{A})$ iff for each a in $\bigcap X$, $F(a) = \mathcal{M}(\mathcal{A})$ iff $\bigcap X \subseteq \bigcap \mathcal{M}(\mathcal{A})$. Since $\bigcap X \supseteq \bigcap \mathcal{M}(\mathcal{A})$ always holds we get the needed equality.$\diamond$.

If $S \subseteq A$, the set

$$\{M |\ M \in \bigcap \mathcal{M}(\mathcal{A}), M \supseteq S\}$$

is called *hull* of S, while the set

$$\bigcap X$$

is called *kernel* of X. So, the closure of X is the hull of the kernel of X. That's why Zariski topology is sometimes called the *hull-kernel topology*.

Proposition 8.5: $\mathcal{M}(\mathcal{A})$ is a T_1 space.

Proof:

$$\overline{\{M\}} = \{M^{'} |\ M^{'} \in \mathcal{M}(\mathcal{A}), M^{'} \supseteq \bigcap \{M\}\} =$$

$$\{M^{'} |\ M^{'} \in \mathcal{M}(\mathcal{A}), M^{'} \supseteq \{M\}\} = \{M\}. \qquad \diamond$$

Proposition 8.6: $\mathcal{M}(\mathcal{A})$ is a T_2 space iff for each pair $M, M^{'}$ of distinct maximal ideals of A there exist $a, a^{'}$ in A, such that, $a \notin M$, $a^{'} \notin M^{'}$ and $aa^{'} \in \bigcap \mathcal{M}(\mathcal{A})$.

Proof: Recall that a space Y is T_2 iff for each $y_1, y_2 \in Y$ there exist F_1, F_2 closed in Y such that $y_1 \notin F_1$, $y_2 \notin F_2$ and $F_1 \cup F_2 = Y$.
($\Rightarrow$): Suppose that for a pair $M, M^{'}$ of distinct maximal ideals of A there exist $F, F^{'}$ closed in $\mathcal{M}(\mathcal{A})$ such that $M \notin F$, $M^{'} \notin F^{'}$ and $F \cup F^{'} = \mathcal{M}(\mathcal{A})$. Equalities

$$F = \bigcap_{b \in B} F(b), \quad F^{'} = \bigcap_{d \in D} F(d)$$

show that $B \subseteq M$, $D \subseteq M^{'}$ and $B \cup D \subseteq M^{''}$, for each $M^{''} \in \mathcal{M}(\mathcal{A})$. There exist though b in B such that $b \notin D$ and $d \in D$ such that $d \notin B$, therefore, $b \in M$ and $b \notin M^{'}$, while $d \in M^{'}$ and $d \notin M$. But, $bd \in \bigcap \mathcal{M}(\mathcal{A})$, since $F(bd) = F(b) \cup F(d) = \mathcal{M}(\mathcal{A})$.
($\Leftarrow$): If $aa^{'} \in \bigcap \mathcal{M}(\mathcal{A})$ then $aa^{'} \in M$ and $aa^{'} \in M^{'}$, hence, $a^{'} \in M$ and $a \in M^{'}$, since every maximal ideal is also prime. Thus, $M \notin F(a)$ and $M^{'} \notin F(a^{'})$, but $F(a) \cup F(a^{'}) = F(aa^{'}) = \mathcal{M}(\mathcal{A})$.$\diamond$

Space $\mathcal{M}(\mathbb{Z})$ with the Zariski topology is not T_2, since $\mathbb{Z}$ is a semi-simple ring i.e.,

$$\bigcap \mathcal{M}(\mathbb{Z}) = \{0\}$$

and an integral domain. Thus, it cannot be T_2 ($a \notin M$, $a^{'} \notin M^{'}$, but $aa^{'} \in \bigcap \mathcal{M}(\mathbb{Z}) = \{0\}$, thus $a = 0$ or $a^{'} = 0$, therefore, $a \in M$ or $a^{'} \in M^{'}$, reaching a contradiction).

Proposition 8.7: Suppose $B \subseteq A$. Then,

$$\bigcap (F(b))_{b\in B} = \emptyset \Leftrightarrow J(B) = A,$$

where $J(B)$ is the ideal generated by B. Moreover, if $J(B) = A$, then there exists a finite subfamily of $(F(b))_{b\in B}$ with a finite intersection too.

Proof: ($\Rightarrow$): Since,

$$\bigcap_{b\in B} F(b) = \{M|\ M \in \mathcal{M}(\mathcal{A}),\ M \supseteq B\} = \emptyset,$$

$J(B) = A$, otherwise $J(B)$ would be contained in a maximal ideal by Zorn's lemma.
($\Leftarrow$): It is trivial.
If $J(B) = A$, then $1 \in J(B)$ i.e., there exist $r_1, ..., r_n \in A$ and $b_1, ..., b_n \in B$ such that $1 = r_1 b_1 + ... + r_n b_n$, since $\mathcal{A}$ is a commutative ring with a unit. Then

$$J(\{b_1, ..., b_n\}) = A \quad \Rightarrow \quad \bigcap_{i=1}^{n} F(b_i) = \emptyset. \quad \diamond$$

Proposition 8.8: Every family of closed sets in $\mathcal{M}(\mathcal{A})$, satisfying the finite intersection property, has a non-empty intersection. In other words, $\mathcal{M}(\mathcal{A})$ with the Zariski topology is a compact space.

Proof: Suppose $(F_i)_{i\in I}$ is a family of closed sets in $\mathcal{M}(\mathcal{A})$. For each $i \in I$,

$$F_i = \bigcap_{b\in B_i} F(b).$$

By hypothesis $(F_i)_{i\in I}$ satisfies the finite intersection property. Suppose $\bigcap_{i\in I} F_i = \emptyset$. If we consider the family

$$(F(b))_{b\in B_i,\ i\in I},$$

then

$$\bigcap_{b\in B_i,\ i\in I} F(b) = \emptyset.$$

Then, by Proposition 8.7, there exists a subfamily $F(b_{i_1}), ..., F(b_{i_n})$, with $b_{i_1} \in B_{i_1}, ..., b_{i_n} \in B_{i_n}$ such that

$$\bigcap_{j=1}^{n} F(b_{i_j}) = \emptyset.$$

But,

$$\bigcap_{j=1}^{n} F(b_{i_j}) \supseteq \bigcap_{j=1}^{n} F_{i_j} = \emptyset,$$

which is absurd, since the family $(F_i)_{i\in I}$ satisfies the finite intersection property.$\diamond$

It now arises the natural question on the relation between the w^*-topology initially defined on $\mathcal{M}(\mathcal{A})$, where $\mathcal{A}$ is a complex commutative Banach algebra, and the Zariski topology defined on $\mathcal{M}(\mathcal{A})$.
Consider the abstract basic closed set of the Zarisiki topology

$$F(a) = \{M|\ M \in \mathcal{M}(\mathcal{A}),\ a \in M\} = \{\varphi|\ \varphi \in \Phi(\mathcal{A}),\ \varphi(a) = 0\}.$$

$F(a)$ though is closed in w^*-topology, since, if $(\varphi_i)_{i\in I}$ is a net in $F(a)$ and $\varphi_i \to \varphi$, then

$$\varphi_i(a) \to \varphi(a) \Leftrightarrow \varphi(a) = 0.$$

Thus, $\varphi \in F(a)$, and the Zariski topology is weaker than the w^*-topology.
In special cases the two topologies are equal. As an example of this we prove the following proposition.

Proposition 8.9: If the complex commutative Banach algebra in question is $\mathcal{A} = C(X)$, where X is a compact T_2 space, then the Zariski topology and the w^*-topology on $\mathcal{M}(\mathcal{A})$ are equal.

Proof: We shall establish a homeomorphism between X and $\Phi(C(X))$ equipped with both topologies, therefore $\mathcal{M}(C(X))$, or $\Phi(C(X))$, with the Zariski topology is homeomorphic to $\Phi(C(X))$ with the w^*-topology.
Consider first $\Phi(C(X))$ with the w^*-topology. We define the following mapping

$$\varphi : X \to \Phi(C(X))$$

$$p \mapsto \varphi_p,$$

where φ_p is a *fixed* character of $C(X)$ i.e.,

$$\varphi_p(f) = f(p),$$

for each $f \in C(X)$. φ is $1-1$ and onto $\Phi(C(X))$, since all characters on $C(X)$ are fixed, if X is a compact T_2 space. Also, φ is continuous, since, if $p_i \to p$ then $f(p_i) \to f(p)$, for each f in $C(X)$, iff $\varphi_{p_i}(f) \to \varphi_p(f)$, for each f in $C(X)$, iff $\varphi_{p_i} \to \varphi_p$, in the w^*-topology. Since φ is a continuous $1-1$ function from a compact space (X) onto a T_2 space $(\Phi(C(X)))$, it is a homeomorphism.
We consider now $\Phi(C(X))$ equipped with the Zariski topology. We define again $\varphi : X \to \Phi(C(X))$, $p \mapsto \varphi_p$. Then,

$$\varphi^{-1}(F(f)) = \varphi^{-1}(\{\varphi_p |\ \varphi_p(f) = f(p) = 0\})$$

$$= \{p|\ p \in X, f(p) = 0\} = f^{-1}(\{0\}).$$

Hence, φ is a continuous function. In order to be a homeomorphism we have to prove that the Zariski topology on $\Phi(C(X))$ turns it to a T_2 space. By Proposition 8.6 it suffices to show that for each pair M_r, M_q of distinct maximal ideals, $r \neq q$, such that, there exist f, g in $C(X)$, such that, $f \notin M_r$, $g \notin M_q$ and

$$fg \in \bigcap_{p\in X} M_p$$

$$\Leftrightarrow (fg)(p) = 0,\ \forall p \in X \Leftrightarrow fg = 0.$$

We see that if X is a compact T_2 space, $C(X)$ is semi-simple, since

$$\bigcap_{p\in X} M_p = \bigcap \mathcal{M}(C(X)) = \{0\}.$$

Since X is T_2 there exist open sets G_p, G_q such that $p \in G_p$, $q \in G_q$ and $G_p \cap G_q = \emptyset$. But X is also a $T_{3\frac{1}{2}}$, therefore there exist continuous functions $f, g : X \to [0,1]$, such that

$$f(q) = f(X - G_p) = 0, \qquad f(p) = 1,$$
$$g(p) = g(X - G_q) = 0, \qquad g(q) = 1.$$

Obviously, $fg = 0$.$\diamond$

Next, we give an example of an algebra in which the Zariski topology is strictly weaker than the w^*-topology.

Proposition 8.10: $H(D)$ equipped with the Zariski topology is not T_2, therefore it cannot be equal to the w^*-topology, which is T_2. Hence, since the Zariski topology is always weaker than the w^*-topology, the Zariski topology in this case is strictly weaker than the w^*-topology.

Proof: We show that $H(D)$ is an integral domain i.e.,

$$ab = 0 \;\Rightarrow\; a = 0 \;\vee\; b = 0$$

and semi-simple i.e.,

$$\bigcap \mathcal{M}(H(D)) = \{0\}.$$

Then, by Proposition 8.6, $H(D)$ cannot be T_2, since if for each pair $M, M^{'}$ of distinct maximal ideals of $H(D)$ there exist $a, a^{'}$ in $H(D)$, such that, $a \notin M$, $a^{'} \notin M^{'}$, then it cannot be the case that $aa^{'} \in \bigcap \mathcal{M}(\mathcal{H}(\mathcal{D}))$, because $a \notin M$, $a^{'} \notin M^{'}$ mean that $a, a^{'} \neq 0$, while the semisimplicity of $H(D)$ implies that if $aa^{'} \in \bigcap \mathcal{M}(\mathcal{H}(\mathcal{D}))$, then, by the integral domain property of $H(D)$, $aa^{'} = 0 \;\Rightarrow\; a = 0 \;\vee a^{'} = 0$, which is, of course, absurd.

<u>$H(D)$ is an integral domain:</u> By the Identity Principle, if $\Omega \subseteq \mathbb{C}$ is open and connected, $f : \Omega \to \mathbb{C}$ holomorphic and

$$Z(f) = \{z |\; z \in \Omega, f(z) = 0\}$$

has an accumulation point in Ω, then $f = 0$.
Then, if $f : \Omega \to \mathbb{C}$ holomorphic and $f \neq 0$, then $Z(f)$ is a discrete space and countable. Suppose $f, g \in H(D)$. If $f, g \neq 0$, then $f^\circ, g^\circ \neq 0$, where $f^\circ = f|D^\circ$, since $D = D^{\circ^-}$. If $fg = 0$, then $f^\circ g^\circ = 0$. I.e., $Z(f^\circ g^\circ) = D^\circ$, or $Z(f^\circ) \cup Z(g^\circ) = D^\circ$, which is absurd, since $Z(f^\circ), Z(g^\circ)$ are countable. Therefore, $f^\circ = 0 \;\vee\; g^\circ = 0$, hence $f = 0 \;\vee\; g = 0$.
<u>$H(D)$ is semi-simple:</u> $H(D)$ can be seen as a subalgebra of $C(\partial D)$. ∂D is a compact T_2 space, hence its space of maximal ideals is the space of its fixed maximal ideals,

$$\mathcal{M}(C(\partial D)) = \{M_d |\; d \in \partial D\},$$

where

$$M_d = \{f |\; f \in C(\partial D),\; f(d) = 0\}.$$

It is clear that the sets

$$M_d \cap H(D) = \{f |\; f \in H(D),\; f(d) = 0\}$$

are the fixed maximal ideals of $H(D)$. Later we shall show that these are exactly the maximal ideals of $H(D)$. But,

$$\bigcap_{d\in\partial D}(M_d\cap H(D)) = (\bigcap_{d\in\partial D} M_d)\cap H(D) = \{0\}.$$

Since,

$$\bigcap_{d\in\partial D}(M_d\cap H(D)) \supseteq \bigcap \mathcal{M}(H(D)),$$

$H(D)$ is semi-simple.$\diamond$

If $\mathcal{A}$ is a commutative complex Banach algebra, $\mathcal{A}$ is called *completely regular* iff the Zariski topology on $\mathcal{M}(\mathcal{A})$ equals the w^*-topology on $\mathcal{M}(\mathcal{A})$. We prove now the following simple characterization of a completely regular algebra.

Proposition 8.11: If $\mathcal{A}$ is a commutative complex Banach algebra, then $\mathcal{A}$ is completely regular iff the Zariski topology on $\mathcal{M}(\mathcal{A})$ is T_2.

Proof: ($\Rightarrow$): Trivially, since the w^*-topology is T_2.
($\Leftarrow$): We consider the identity map

$$id : (\mathcal{M}(\mathcal{A}), w^*) \to (\mathcal{M}(\mathcal{A}), Z)$$

$$\varphi \mapsto \varphi,$$

where Z denotes the Zariski topology on $\mathcal{M}(\mathcal{A})$. Since,

$$Z \subseteq w^*,$$

id is continuous, $1-1$ and onto the T_2 space, by hypothesis, $(\mathcal{M}(\mathcal{A}), Z)$. Since $(\mathcal{M}(\mathcal{A}), w^*)$ is compact, id is a homeomorphism. Hence, id is open and $w^* \subseteq Z$. Therefore,

$$w^* = Z. \quad \diamond$$

We see now how some of the above results apply on Boolean algebras.

A *Boolean algebra* $\mathcal{B}$ is a complemented distributive lattice. In a Boolean algebra the complement of ana element is unique. A Boolean ring $\mathcal{R}$ is a ring with a unit in which each element is *idempotent* i.e.,

$$a^2 = a, \qquad \forall a \in R.$$

It is easy to see that in a Boolean ring

$$a + a = 0,$$
$$ab = ba.$$

There is also a natural correspondence between Boolean algebras and Boolean rings (isomorphic categories).
Many results of the theory of Boolean algebras can be seen as special cases of the theory of Banach algebras. E.g., the following proposition can be proved trivially.

Proposition 8.12 (Mazur theorem for Boolean rings): If a Boolean ring is a division ring (therefore, a field), then it is isomorphic to $\{0, 1\}$.

Since a Boolean ring is a commutative ring with a unit, we can study its structure space.

Proposition 8.13: If $\mathcal{R}$ is a Boolean ring, then

$$M \in \mathcal{M}(\mathcal{R}) \Leftrightarrow R/M \cong \{0, 1\}.$$

Proof: The ($\Rightarrow$) direction uses the Mazur theorem for Boolean rings, while the ($\Leftarrow$) direction is trivial, since $\{0, 1\}$ is a field.$\diamond$

By Proposition 8.13 in the theory of Boolean algebras the concepts of maximal ideal and $\{0, 1\}$-ideal are identical.

A character φ of a Boolean ring $\mathcal{R}$ is a homomorphism of Boolean rings

$$\varphi : R \to \{0, 1\}.$$

Clearly, a character is an epimorphism. Since there is a unique isomorphism between R/M and $\{0, 1\}$, the mapping

$$Ker : \Phi(\mathcal{R}) \to \mathcal{M}(\mathcal{R})$$

expresses the natural identification between the characters of $\mathcal{R}$ and its maximal $\{0, 1\}$-ideals.

We may view a Boolean ring as an algebra over $\{0, 1\}$ and then

$$sp(0) = \{0\}, \quad sp(1) = \{1\}, \quad sp(a) = \{0, 1\}, \ \forall a \notin \{0, 1\}.$$

Obviously,

$$\varphi(a) \in sp(a)$$

and since for a Boolean ring $\mathcal{R}$ the following are equivalent:

(i) $a \neq 1$.
(ii) The ideal generated by a is proper.
(iii) There exists a maximal ideal which contains a.
(iv) There exists a character φ of $\mathcal{R}$ such that $\varphi(a) = 0$,

then

$$sp(a) = \{\varphi(a) | \ \varphi \in \Phi(\mathcal{R})\}.$$

If $\mathcal{R}$ is considered with the discrete norm, φ is continuous and we may define on

$$\prod_{a \in R} sp(a)$$

the product topology, or equivalently the w^*-topology and then $\mathcal{M}(\mathcal{R})$ is a closed subset of $\prod_{a \in R} sp(a)$, therefore, it is a compact T_2 space.

On $\mathcal{M}(\mathcal{R})$ we may define the Zariski topology Z.

Proposition 8.14: If $\mathcal{R}$ is a Boolean ring, then $(\mathcal{M}(\mathcal{R}), Z)$ is a T_2 space.

Proof: Obviously

$$M \in \mathcal{M}(\mathcal{R}) \Leftrightarrow \forall a \in R, \ a \in M \ \veebar \ a^{'} \in M,$$

where $a^{'}$ is the complement of a. Then the criterion of Proposition 8.6 is trivially satisfied.$\diamond$

Hence, the identity map

$$id : (\mathcal{M}(\mathcal{R}), w^*) \to (\mathcal{M}(\mathcal{R}), Z)$$

$$\varphi \mapsto \varphi$$

is a homeomorphism i.e.,

$$w^* = Z.$$

In the next section we shall study the analogy between the Gelfand transform and the Stone transform.

9 The Gelfand transform

If $\mathcal{A}$ is complex commutative Banach algebra, the *Gelfand transform* of $\mathcal{A}$ is the mapping

$$G : A \to C(\mathcal{M}(\mathcal{A}))$$

$$a \mapsto \hat{a},$$

where

$$\hat{a}(\varphi) = \varphi(a).$$

$\mathcal{M}(\mathcal{A})$ is equipped with the w^*-topology and by its definition it is obvious that $\hat{a} \in C(\mathcal{M}(\mathcal{A}))$. G is, obviously, a homomorphism of $\mathbb{C}$-algebras. Also

$$||\hat{a}||_\infty = sup\{|\varphi(a)| \mid \varphi \in \mathcal{M}(\mathcal{A})\}.$$

Therefore, G is continuous.
The Gelfand transform is analogous to the *Alaoglou transform*

$$A : X \to C(S_{X^*}, w^*)$$

$$x \mapsto \hat{x},$$

where X is a normed space and

$$\hat{x}(x^*) = x^*(x).$$

The Alaoglou transform is directly connected to the natural imbedding of X into X^{**}, and it is a linear isometric imbedding. A expresses the fact that a normed space is imbedded in $C(K)$, where K is a compact Hausdorff space.
The analogy is stronger between the Gelfand transform and the Stone transform. If $\mathcal{R}$ is a Boolean ring, the *Stone transform* of $\mathcal{R}$ is the mapping

$$S : A \to C(\mathcal{M}(\mathcal{R}))$$

$$a \mapsto \hat{a},$$

where

$$\hat{a}(\varphi) = \varphi(a),$$

and $C(\mathcal{M}(\mathcal{R}))$ is $C(\mathcal{M}(\mathcal{R}), 2)$. If we consider $\mathcal{R}$ as an algebra over $2 = \{0, 1\}$ with the discrete norm, then the Stone transform is also an isometry.
If $\mathcal{A}$ is a commutative algebra, the *radical* of $\mathcal{A}$ is

$$Rad\mathcal{A} = \bigcap \mathcal{M}(\mathcal{A}).$$

Obviously, $\mathcal{A}$ is semi-simple iff $Rad\mathcal{A} = \{0\}$.
If $\mathcal{A}$ is a complex commutative Banach algebra, then

$$kerG = Rad\mathcal{A}.$$

Obviously, G is $1-1$ iff $\mathcal{A}$ is semi-simple.
We examine now when G is a homeomorphic imbedding i.e., G is $1-1$ and G^{-1} is also

continuous.

Proposition 9.1: The following are equivalent:
(i) G is a homeomorphic imbedding.
(ii) G is $1-1$ and $G(A)$ is closed.
(iii) G is $1-1$ and $G(A)$ is a Banach space.
(iv) The mapping

$$r : A \to \mathbb{R}$$

$$a \mapsto r(a),$$

where $r(a)$ is the spectral radius of a, is a norm on A, equivalent to its norm $||.||$.
(v) r is a complete norm on A.
(vi) $\forall a \in A,\ \exists M > 0 : \quad ||a|| \leq Mr(a)$.
(vii) $\forall a \in A,\ \exists K > 0 : \quad ||a||^2 \leq K||a^2||$.

Proof: ((i)$\Rightarrow$(ii)) is trivial.
(ii)$\Rightarrow$(iii) is also trivial since $C(\mathcal{M}(\mathcal{A}))$ is a Banach algebra.
(iii)$\Rightarrow$(i): By the open mapping theorem.
(i)$\Rightarrow$(iv): It is obvious, since $r(a) = ||\hat{a}||$ and by the continuity of G^{-1} we take the unknown inequality for the equivalence of the two norms.
(iv)$\Rightarrow$(v): It is obvious, since completeness is preserved when the space is a Banach space.
(v)$\Rightarrow$(i): G is $1-1$, since r is a norm, and G^{-1} is continuous by the open mapping theorem.
(iv)$\Rightarrow$(vi) is trivial.
(vi)$\Rightarrow$(vii): By Proposition 6.1 we get

$$||a|| \leq M||a^2||^{\frac{1}{2}} \Rightarrow ||a||^2 \leq M^2||a^2||$$

i.e., $K = M^2$.
(vii)$\Rightarrow$(vi): We can easily see that

$$||a||^{2^n} \leq K^{2^n-1}||a^{2^n}|| \Rightarrow ||a|| \leq K^{\frac{2^n-1}{2^n}}||a^{2^n}||^{\frac{1}{2^n}}.$$

Therefore,

$$||a|| \leq Kr(a).$$

(vi)$\Rightarrow$(iv): Obviously, r is a norm and it is equivalent to $||.||$.$\diamond$

Proposition 9.2: G is an isometric imbedding iff

$$||a^2|| = ||a||^2, \quad \forall a \in A.$$

Proof: ($\Rightarrow$) is obvious, since the above property holds in $C(\mathcal{M}(\mathcal{A}))$.
($\Leftarrow$): We can easily see that

$$||a^{2^n}|| = ||a||^{2^n}.$$

Therefore,

$$||\hat{a}|| = r(a) = lim||a^n||^{\frac{1}{n}} = lim||a^{2^n}||^{\frac{1}{2^n}} = ||a||. \quad \diamond$$

We include here without proof the following classical theorems:

The real Stone-Weierstrass Theorem (RSW): If X is a compact space and $\mathcal{A}$ a subalgebra of $C(X, \mathbb{R})$ which separates its points i.e.,

$$\forall x, y \in X,\ x \neq y \Rightarrow \exists f \in A:\ f(x) \neq f(y),$$

then $\mathcal{A}$ is dense in $C(X, \mathbb{R})$ with respect to $||.||_\infty$.

The complex Stone-Weierstrass Theorem (CSW): If X is a compact space and $\mathcal{A}$ a subalgebra of $C(X, \mathbb{C})$ which separates its points and it is closed with respect to $*$, then $\mathcal{A}$ is dense in $C(X, \mathbb{C})$ with respect to $||.||_\infty$.

Let $\mathcal{A}$ be a complex commutative Banach algebra. Since G is a homomorphism of algebras $G(A)$ is a subalgebra of $C(\Phi(A))$. Also, $G(A)$ separates points, since if $\varphi, \psi \in \Phi(A)$ and $\varphi \neq \psi$, then there is an element a of A such that $\varphi(a) \neq \psi(a)$ iff $\hat{a}(\varphi) \neq \hat{a}(\psi)$. In order to apply the complex case of Stone-Weierstrass theorem we need $G(A)$ to be closed with respect to $*$. That is satisfied if A is $*$-algebra and G preserves $*$. Next proposition shows when G preserves $*$.

Proposition 9.3: Let $\mathcal{A}$ be a complex commutative Banach $*$-algebra. The following are equivalent:
(i) G preserves $*$ i.e., $\hat{a^*} = \hat{a}^*$, for each a in A.
(ii) For each φ in $\Phi(A)$ and a in A, $\varphi(a^*) = \overline{\varphi(a)}$.
(iii) For each φ in $\Phi(A)$ and a in $Her(A)$, $\varphi(a) \in \mathbb{R}$.
(iv) For each a in $Her(A)$, $sp(a) \subseteq \mathbb{R}$ i.e., $\mathcal{A}$ is hermitian.
For each a in A $1 + a^*a$ is invertible i.e., $\mathcal{A}$ is symmetric.
If one (i)-(v) holds, then $G(A)$ is $*$-closed, therefore

$$\overline{G(A)} = C(\mathcal{C}(\mathcal{A})).$$

Proof: The equivalence between (i) and (ii) is obvious. Likewise for the implication $(ii) \Rightarrow (iii)$. For the inverse consider a φ in $\mathcal{M}(\mathcal{A})$ and an a in A. Then, $a = a_1 + ia_2,\ a_1, a_2 \in Her(A)$ and

$$\varphi(a^*) = \varphi(a_1) - i\varphi(a_2) = \overline{\varphi(a_1) + i\varphi(a_2)} = \overline{\varphi(a)}.$$

The equivalence between (iii) and (iv) derives from the fact that $sp(a) = \{\varphi(a) \mid \varphi \in \mathcal{M}(\mathcal{A})\}$ for the algebra $\mathcal{A}$ of our hypothesis. Implication $(v) \Rightarrow (iv)$ is Proposition 7.13, while the inverse expresses the fact that if $\mathcal{A}$ is a complex commutative and hermitian Banach $*$-algebra it is also it is also symmetric. We show this through (ii) i.e., $(ii) \Rightarrow ((v)$:

$$\varphi(1 + a^*a) = 1 + |\varphi(a)|^2 > 0,$$

for each φ i.e., $1 + a^*a$ does not belong to any maximal ideal, therefore it is invertible.

Proposition 9.4: Let $\mathcal{A}$ be a complex commutative Banach algebra. Then G is an isometrical isomorphism onto $C(\mathcal{M}(\mathcal{A}))$ iff there is $* : A \to A$ which turns $\mathcal{A}$ into a C^*-algebra (then G also preserves $*$).

Proof: $(\Rightarrow)$ Obviously, this property holds on $C(\mathcal{M}(\mathcal{A}))$.
$(\Leftarrow)$ Since $\mathcal{A}$ is commutative, each element of A is normal. Then, by Proposition 3.1, $||a^2|| = ||a||^2$ for each a. Thus, G is an isometrical embedding, hence $G(A)$ is closed in $C(\mathcal{M}(\mathcal{A}))$. By Proposition 7.8 $\mathcal{A}$ is hermitian. Therefore, by Proposition 9.3,

$$G(A) = \overline{G(A)} = C\mathcal{M}(\mathcal{A}). \quad \diamond$$

Thus, we have arrived to the desired representation theorem.

Proposition 9.5 (Commutative Gelfand-Naimark theorem): A complex Banach algebra $\mathcal{A}$ is isometrically isomorphic to the algebra $C(K, \mathbb{C})$, for some compact Hausdorff space K, if and only if it is commutative and there is an involution defined on $\mathcal{A}$ which turns it to a C^*-algebra.

Using the following theorem of Gelfand and Kolmogorov (1939):

If X, Y are compact T_2 spaces

$$C(X, \mathbb{C}) \cong C(Y, \mathbb{C}) \Rightarrow X \cong Y$$

i.e., the isomorphism of the rings of continuous functions implies the homeomorphism of the corresponding spaces.

we see that the compact T_2 space K is unique up to homeomorphism. Hence, we can take K to be $\mathcal{M}(\mathcal{A})$ and G as the isometric isomorphism, which also preserves $*$.

We may formulate Proposition 9.5 in the following form.

Proposition 9.6: A complex Banach algebra $\mathcal{A}$ is isometrically isomorphic to the algebra $C(K, \mathbb{C})$, for some compact Hausdorff space K, if and only if it is commutative and there is a norm and an involution defined on $\mathcal{A}$ which turn $\mathcal{A}$ into a C^*-algebra.

The above formulation is purely algebraic, since we do not refer to a norm given from the beginning, but to the existence of an appropriate norm.
Let $\mathcal{A}$ be an algebra satisfying the above conditions. If $||.||$ is a norm satisfying the desired properties, then for each a

$$||a|| = sup\{|\varphi(a)| \mid \varphi \in \mathcal{M}(\mathcal{A})\}.$$

I.e., $||.||$ is uniquely determined by the algebraic structure, since $\mathcal{M}(\mathcal{A})$ is defined algebraically. Also, if $*$ is an appropriate involution, then for each a and each φ

$$\varphi(a^*) = \overline{\varphi(a)}.$$

This equality uniquely determines $*$, since $\mathcal{A}$ is semi-simple i.e., $*$ is also determined by the algebraic structure.
Both of these remarks show that if $T : A \to C(K, \mathbb{C})$ is an isomorphism of C^*-algebras, than T preserves the norm (i.e., T is an isometry) and the involution $*$. I.e., with algebraic hypotheses on the algebras $\mathcal{A}$ and $C(K, \mathbb{C})$ we get strong analytical results for the C^*-algebras $\mathcal{A}$ and $C(K, \mathbb{C})$.

As an application of the Gelfan-Naimark theorem we prove the existence of the Stone-Cech compactification of a topological space.

If X is a topological space, a *free compact T_2 space on X* is a pair (K, e), where K is a compact T_2 space, $e : X \to K$ continuous, and for each f in $C_b(X)$ there is a unique $\tilde{f}$ in $C(K)$ such that

$$\tilde{f} \circ e = f.$$

We give without proof the following proposition.

Proposition 9.6: If (K, e) is a free compact T_2 space on X, then:
(i) $\overline{e(X)} = K$.

(ii) (K, e) is essentially unique.
(iii) (K, e) is the Stone-Cech compactification of X i.e., e is a homeomorphic embedding iff X is $T_{3\frac{1}{2}}$ and T_1.

We shall now construct the above pair (K, e). We consider

$$K = \mathcal{M}(C_b(X))$$

and $e : X \to K$ such that $x \mapsto e_x$, where

$$e_x(f) = f(x).$$

By the way w^* is defined on $\mathcal{M}(C_b(X))$ e is continuous. The unique object having the property of $\tilde{f}$ is $G(f)$, where G is the Gelfand transform

$$G : C_b(X) \to C(K) \qquad f \mapsto \hat{f}.$$

Trivially,

$$\hat{f} \circ e = f.$$

As it is known $C_b(X)$ is a commutative C^*-algebra. Therefore, G is is an isometric embedding onto $C(K)$. The uniqueness of $\hat{f}$ is now derived directly as follows.
Suppose $\tilde{h}$ in $C(K)$ such that $\tilde{h} \circ e = f$. By the above, there is h in $C_b(X)$ such that

$$\tilde{h} = \hat{h},$$

and

$$\hat{h} \circ e = h = f \Rightarrow \hat{h} = \hat{f} \Rightarrow \tilde{h} = \hat{f}.$$

It is easy to see that e is a homeomorphism iff X is compact and T_2 space.

10 Wiener's lemma

As we have already said in Paragraph 4, $L^1(\mathbb{Z})$ is the group algebra of functions $f : \mathbb{Z} \to \mathbb{C}$ such that

$$\sum_{k\in\mathbb{Z}} |f(k)| < \infty.$$

$L^1(\mathbb{Z})$ is a commutative Banach algebra with norm

$$\|f\| = \sum_{k\in\mathbb{Z}} |f(k)|$$

and multiplication the convolution operation

$$f * g(n) = \sum_{k\in\mathbb{Z}} f(n-k)g(k).$$

The multiplication unit is $1 = \chi_{\{0\}} = \omega_0$, the characteristic function of $\{0\}$, and it is clear that $\|\omega_0\| = 1$. To see that $f * g \in L^1(\mathbb{Z})$ we use a discrete form of Fubini's theorem:

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} |(f*g)(n)| &= \sum_{n\in\mathbb{Z}} |\sum_{k\in\mathbb{Z}} f(n-k)g(k)| \\
&\leq \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |f(n-k)||g(k)| \\
&= \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |f(n-k)||g(k)| \\
&= \sum_{k\in\mathbb{Z}} |g(k)| \sum_{n\in\mathbb{Z}} |f(n-k)| \\
&= \sum_{k\in\mathbb{Z}} |g(k)| \; \|f\| = \|f\| \; \|g\|.
\end{aligned}$$

Since $f * g \in L^1(\mathbb{Z})$ is established, the above result expresses the validity of the normed algebra inequality

$$\|f * g\| \leq \|f\| \; \|g\|$$

of $L^1(\mathbb{Z})$.

Obviously, every function on $\mathbb{Z}$ with finitely many values is in $L^1(\mathbb{Z})$. The elements ω_n of $L^1(\mathbb{Z})$, where ω_n is the characteristic function of $n \in \mathbb{Z}$, satisfy the following obvious properties:

(i) $\omega_n = \omega_1^n = \underbrace{\omega_1 * \omega_1 * \ldots * \omega_1}_{n}$.

(ii) If $f \in L^1(\mathbb{Z})$, then

$$f = \sum_{k\in\mathbb{Z}} f(k)\omega_k.$$

(iii) $\|\omega_k\| = 1$.

(iv) $\omega_n * \omega_k = \omega_{n+k}$.

We prove now an equivalent description of $\mathcal{M}_{L^1(\mathbb{Z})}$ in order to use it in the Gelfand transform on $L^1(\mathbb{Z})$.

Proposition 10.1: The maximal ideal space of the Banach algebra $L^1(\mathbb{Z})$ is homeomorphic to the unit circle $\mathbb{T}$.

Proof: We define the following mapping $\Phi : \mathbb{T} \to \mathcal{M}_{L^1(\mathbb{Z})}$

$$z \mapsto \varphi_z, \qquad \varphi_z : L^1(\mathbb{Z}) \to \mathbb{C},$$

where

$$\varphi_z(f) = \sum_{k\in\mathbb{Z}} f(k)z^k.$$

Trivially, φ_z is linear. It is also bounded, since

$$|\varphi_z(f)| = |\sum_{k\in\mathbb{Z}} f(k)z^k| \leq \sum_{k\in\mathbb{Z}} |f(k)|\ |z^k| = \sum_{k\in\mathbb{Z}} |f(k)| < \infty$$

and non-trivial, since $\varphi_z(1) = \varphi_z(\omega_0) = z^0 = 1$. Using Fubini again, we show that φ_z is actually an algebra homomorphism.

$$\varphi_z(f * g) = \sum_{k\in\mathbb{Z}} (f * g)(k)z^k = \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} f(k-n)g(n)z^k$$

$$= \sum_{k\in\mathbb{Z}} z^k \sum_{n\in\mathbb{Z}} f(k-n)g(n) = \sum_{k\in\mathbb{Z}} z^n z^{k-n} \sum_{n\in\mathbb{Z}} f(k-n)g(n)$$

$$= \sum_{n\in\mathbb{Z}} z^n g(n) \sum_{k\in\mathbb{Z}} f(k-n)z^{k-n} = \varphi_z(g)\varphi_z(f).$$

Φ is 1-1, since $\varphi_{z_1} = \varphi_{z_2} \to \varphi_{z_1}(\omega_1) = \varphi_{z_2}(\omega_1) \leftrightarrow z_1 = z_2$, since

$$\varphi_z(\omega_1) = \sum_{k\in\mathbb{Z}} \omega_1(k)z^k = z^1 = z.$$

Φ is onto $\mathcal{M}_{L^1(\mathbb{Z})}$, since for each $\varphi \in \mathcal{M}_{L^1(\mathbb{Z})}$ we can find $z_0 \in \mathbb{T}$, such that $\varphi(f) = \varphi_{z_0}(f)$, $\forall f \in L^1(\mathbb{Z})$. We find this z_0 using the above properties of ω_k.

$$\varphi(f) = \varphi(\sum_{k\in\mathbb{Z}} f(k)\omega_k) = \sum_{k\in\mathbb{Z}} f(k)\varphi(\omega_k) = \sum_{k\in\mathbb{Z}} f(k)\varphi(\omega_1)^k = \sum_{k\in\mathbb{Z}} f(k)z_0^k = \varphi_{z_0}(f)$$

therefore,

$$z_0 = \varphi(\omega_1).$$

In the above equalities we have used the obvious property $\varphi(\omega_k) = \varphi(\omega_1^k) = \phi(\omega_1)^k$ and the equality

$$\varphi(\sum_{k\in\mathbb{Z}} f(k)\omega_k) = \sum_{k\in\mathbb{Z}} f(k)\varphi(\omega_k),$$

which is justified by a simple continuity argument.

To show that Φ is a homeomorphism it suffices to show that Φ is continuous, since $\mathcal{M}_{L^1(\mathbb{Z})}$ is a compact T_2-space and Φ is already $1-1$ and onto $\mathcal{M}_{L^1(\mathbb{Z})}$. Continuity of Φ is established in a standard way, considering a sequence (z_m) and z in $\mathbb{T}$ such that $z_m \to z$ and showing that $\Phi(z_m) \to \Phi(z)$, observing that $\varphi_z(f) = \sum_{k\in\mathbb{Z}} f(k)z^k$ is the uniform limit in z of the continuous functions $\sum_{k=-N}^{N} f(k)z^k$.$\diamond$

Thus, we may identify $\mathcal{M}_{L^1(\mathbb{Z})}$ with $\mathbb{T}$ and then $C(\mathcal{M}_{L^1(\mathbb{Z})})$ with $C(\mathbb{T})$, and then, the Gelfand transform $\hat{}: L^1(\mathbb{Z}) \to C(\mathcal{M}_{L^1(\mathbb{Z})})$ becomes

$$\hat{}: L^1(\mathbb{Z}) \to C(\mathbb{T}),$$

where

$$\hat{f}(z) = \hat{f}(\varphi_z) = \varphi_z(f) = \sum_{k\in\mathbb{Z}} f(k)z^k, \quad \textit{and} \quad \sum_{k\in\mathbb{Z}} |f(k)| < \infty.$$

$\hat{}$ is not onto $C(\mathbb{T})$, since

$$f(z) = \sum_{n=1}^{\infty} \frac{e^{inlogn}}{n} z^n$$

is absolutely non-convergent.
$\hat{}$ is not $1-1$, since, by Proposition 9.2, it suffices to find an element f of $L^1(\mathbb{Z})$ such that $||f^2|| = ||f * f|| \neq ||f||^2$. Just consider for that the function f which is zero except $f(1) = 1$ and $f(2) = 2$. It is easy to see that if

$$\hat{f}(z) = \sum_{k\in\mathbb{Z}} f(k)z^k,$$

then $f(k)$ are the Fourier coefficients of $\hat{f}$ i.e.,

$$f(k) = \frac{1}{2\pi}\int_0^{2\pi} \hat{f}(e^{it})e^{-ikt}dt.$$

Proposition 10.2 (Wiener's lemma): If f is in $C(\mathbb{T})$ that has an absolutely convergent Fourier series and never vanishes, then its reciprocal $\frac{1}{f}$ has such a Fourier series too.

Proof: By hypothesis there exists g in $L^1(\mathbb{Z})$ such that $f = \hat{g}$. Since $\hat{g} \neq 0$, then $0 \notin sp(g)$. Suppose $0 \in sp(g)$. Then, $0 = \varphi(g)$, for some φ. But $\varphi = \varphi_z$, therefore $0 = \varphi_z(g)$. This contradicts our hypothesis $\hat{g}(z) = \varphi_z(g) \neq 0$.
Thus, g is invertible in $L^1(\mathbb{Z})$, with inverse $\frac{1}{g}$. Then,

$$g * \frac{1}{g} = \omega_0 \Rightarrow \widehat{g * \frac{1}{g}} = \hat{\omega_0} \Rightarrow \hat{g}\hat{\frac{1}{g}} = 1$$

$$\Leftrightarrow f\hat{\frac{1}{g}} = 1 \Rightarrow \hat{\frac{1}{g}} = \frac{1}{f}.$$

Hence, $\frac{1}{f}$ has absolutely convergent Fourier series.$\diamond$

For a classical, more involved, proof Wiener's lemma through Fourier analysis see [Zygmund 1959] pp.245-6.

11 The Continuous Function Calculus (CFC)

The CFC is an application of the commutative Gelfand-Naimark theorem according to which we can talk in a "functional" way about the elements of $C^*(a)$, the C^*-algebra generated by a normal element of a general C^*-algebra $\mathcal{A}$. As we shall see a continuous function on $sp(a)$ determines an element of $C^*(a)$ and each element of $C^*(a)$ is determined by such a function.
If $\mathcal{A}$ is a general (not necessarily commutative) C^*-algebra, then

$$C^*(a) = \overline{\{p(a, a^*) |\ p \in \mathbb{C}[x, y]\}},$$

where a is a normal element of $\mathcal{A}$, is a commutative C^*-algebra, since a, a^* commute. So, we may apply the commutative Gelfand-Naimark theorem on $C^*(a)$

$$\wedge : C^*(a) \to \mathcal{C}(\mathcal{M}_{C^*(a)})$$

where $\mathcal{M}_{C^*(a)}$ is actually $sp(a)$, since the mapping

$$e : \mathcal{M}_{C^*(a)} \to sp(a)$$

$$\varphi \mapsto \varphi(a)$$

is 1-1, onto $sp(a)$ and, therefore, a homeomorphism.
If we define τ to be the inverse function of $\wedge$, then

$$\tau : \mathcal{C}(sp(a)) \to C^*(a)$$

$$\tau(id_{sp(a)}) = a$$

is an isometric $*$-isomorphism. Actually, τ is the unique mapping with the above properties, since an element f of $\mathcal{C}(sp(a))$ with $f(z) = p(z, z^*)$ necessarily goes to $p(a, a^*)$ and each other element f of $\mathcal{C}(sp(a))$ goes to an element of $C^*(a)$, which we are entitled to denote by $f(a)$. Thus,

$$C^*(a) = \{f(a)|\ f \in \mathcal{C}(sp(a))\}$$

So, a is $id_{sp(a)}(a)$, since $\tau(id_{sp(a)}) = a$.

So, a normal element of a general C^*-algebra $\mathcal{A}$ can be seen as a continuous function. In that way, concepts and results on continuous functions are transferred to $\mathcal{A}$. Also, properties of the complex numbers are transferred to functions, since their operations are defined pointwisely. In that way, properties of complex numbers are found in $\mathcal{A}$. This is the spirit of the following two propositions.

Proposition 11.1: If a is a normal element of a general C^*-algebra $\mathcal{A}$, then:

(i) (Spectral mapping theorem) $sp(f(a)) = f(sp(a))$.
(ii) $a^* = a$ (a is hermitian) $\Leftrightarrow sp(a) \subseteq \mathbb{R}$.
(iii) $a^* = a$ and $a^2 = a$ (a is a projection) $\Leftrightarrow sp(a) \subseteq \{0, 1\}$.
(iv) $a^*a = aa^* = 1$ (a is unitary) $\Leftrightarrow sp(a) \subseteq S^1$.

Proof: (i) Since spectrum is algebraically determined, the "same" elements of "same" algebras have the same spectrum. Thus, since $f \mapsto f(a)$, then

$$sp(f(a)) = sp(f) = f(sp(a)).$$

(ii) ($\Rightarrow$) Every C^*-algebra is Hermitian.
($\Leftarrow$) If $sp(a) \subseteq \mathbb{R}$, then

$$id_{sp(a)} = id^*_{sp(a)} \leftrightarrow \tau(id) = \tau(id^*) \leftrightarrow a = a^*$$

(iii) Since

$$a^2 = a \leftrightarrow \tau(id^2) = \tau(id) \leftrightarrow id^2 = id,$$

then for each $\lambda \in sp(a)$

$$id^2(\lambda) = id(\lambda) \leftrightarrow \lambda^2 = \lambda \leftrightarrow \lambda \in \{0, 1\}$$

(iv) Since $a^*a = aa^* = 1$, then $(id^*)(id) = (id)(id^*) = 1$, where 1 is the constant function 1 on $sp(a)$. So, for each $\lambda \in sp(a)$

$$\lambda\lambda^- = \lambda^-\lambda \leftrightarrow |\lambda|^2 = 1 \leftrightarrow |\lambda| = 1.$$

Proposition 11.2: $f(g(a)) = (f \circ g)(a)$

Proof: In the first place, from the spectral mapping theorem the above equality is meaningful. We fix a function g in $\mathcal{C}(sp(a))$. Then we have the following two mappings:

$$\tau : \mathcal{C}(sp(g(a)) \to C^*(g(a))$$

$$\tau(f) = f(g(a))$$

and

$$\tau' : \mathcal{C}(sp(g(a))) \to C^*(a)$$
$$\tau'(f) = (f \circ g)(a)$$

Since τ and τ' are continuous morphisms of $*$-algebras (in fact $C^*(g(a)) \preceq C^*(a)$) and send $id_{sp(g(a))}$ to $g(a)$, then they are identical (uniqueness of the CFC). Therefore,

$$\tau(f) = \tau'(f) \leftrightarrow f(g(a)) = (f \circ g)(a).$$

12 Representations of C^*-algebras

Our final aim is to prove the general representation theorem of C^*-algebras, namely that each C^*-algebra is isomorphic (with respect to all of its structure) to a C^*-subalgebra of $\mathfrak{B}(H)$, for some Hilbert space H.
As we have already explained our C^*-algebra has essentially always a unit, since if it has not one, then the representation of its unitization leads to a representation of the original C^*-algebra. In what follows we give a general description of what follows without proofs.
If $\mathcal{A}$ is a C^*-algebra and H a Hilbert space, a *representation* of $\mathcal{A}$ into H is a mapping

$$\pi : A \to \mathfrak{B}(H)$$

which preserves the algebraic structure (including $*$ and 1). The representation is called *faithful* iff it is $1-1$.
As we prove later, any representation is continuous. Furthermore, if it is faithful, it is an isometry. According to these definition the general Gelfand-Naimark theorem is formulated as follows:

"Any C^*-algebra has a faithful representation in a Hilbert space."

We see first how a representation is constructed in a special and familiar C^*-algebra, $C([0,1])$. We wish to see the elements of $C([0,1])$ as operators on some Hilbert space. It is well known that a continuous function f acts as an operator when it acts multiplicatively i.e.,

$$g \mapsto fg.$$

The Hilbert space closer to $C([0,1])$ is $L^2([0,1])$. We then define

$$\pi : C([0,1]) \to \mathfrak{B}(L^2([0,1])) \qquad f \mapsto \pi(f),$$

where

$$\pi(f)g = fg, \qquad \forall g \in L^2([0,1]).$$

It is easy to see that π is a faithful representation of $C([0,1])$.
The general construction which gives the representation of a general C^*-algebra mimics the above special construction. Although the general construction is far more complicated, it is a natural and necessary extension of the above simple construction.
Let's see how the above method works in the $C(K)$-case, where K is a compact Hausdorff space.
We need to find a Hilbert space which is as close to $C(K)$ as $L^2([0,1])$ is to $[0,1]$. Hence, we need to find a measure on K. But we know that a measure on K essentially is a positive linear form on $C(K)$ i.e., a linear form φ such that

$$f \geq 0 \Rightarrow \varphi(f) \geq 0.$$

To be more specific, if μ is a regular measure on K and if we define

$$\varphi(f) = \int f d\mu,$$

then φ is a positive linear form on $C(K)$. By Riesz theorem the inverse is also true. I.e., if φ is a positive linear form on $C(K)$, then there is a unique regular measure on

K such that $\varphi(f) = \int f d\mu$, for each f in $C(K)$.
We work with positive linear forms instead regular measures since that is the concept which is generalized in general C^*-algebras.
Having a positive linear form φ on $C(K)$ we need to construct the needed Hilbert space. We first note that $L^2([0,1])$ is the completion of $C([0,1])$ with respect to the obvious inner product defined on $C([0,1])$. Thus, we shall try to define an inner product on $C(K)$ and then by completion to get a Hilbert space. If μ is the measure corresponding to φ, the most natural definition is

$$< f, g > = \int f g^* d\mu = \varphi(f g^*).$$

Unfortunately, this operatin is not an inner product, since it is possible that $< f, f > = 0$ and $f \neq 0$. The solution to this problem is standard. We consider the quotient with respect to the set of all problematic elements, here

$$\mathcal{I} = \{f | f \in C(K),\ < f, f > = 0\}.$$

$\mathcal{J}$ is a subspace of $C(K)$ and on the quotient space $C(K)/\mathcal{J}$ the operation

$$< [f], [g] > = < f, g >$$

is well defined and an inner product. We then define

$$L^2(K, \varphi) = \widetilde{C(K)/}\mathcal{J},$$

where $\widetilde{C(K)/}\mathcal{J}$ is the completion of $C(K)/\mathcal{J}$. Thus, a Hilbert space is constructed. In complete analogy to our first simple representation we define

$$\pi : C(K) \to \mathfrak{B}(L^2(K, \varphi)) \qquad f \mapsto \pi(f),$$

where

$$\pi(f)g = fg, \qquad \forall g \in C(K).$$

$\pi(f)$ is well defined, since $\mathcal{J}$ is an ideal of $C(K)$. Here we defined π, not on $L^2(K, \varphi)$, but on its dense subspace $C(K)/\mathcal{J}$. But, since it is continuous, it extends uniquely on $L^2(K, \varphi)$. We can show that π is a representation of $C(K)$, though it is not generally $1-1$, since by taking the quotient we reduced dramatically the size of the constructed Hilbert space H. As a result $C(K)$ may not be embedded into $\mathfrak{B}(H)$ in a $1-1$ way. We discuss the solution to this problem at the end.
If $\mathcal{A}$ is a general C^*- algebra, trying to generalize the construction in the $C(K)$-case we face two problems. First we cannot talk from the beginning about positive linear forms. In order to do that we define a concept of order on $\mathcal{A}$. A second problem is that generally $\mathcal{A}$ is not commutative. Then previous technics applicable in the commutative case have to be transformed.
A definition of an order on $\mathcal{A}$, or of a positive element of $\mathcal{A}$ has to generalize the corresponding definitions in C^*- algebras where there is an order on them. E.g., in $C(K)$

$$f \geq 0 \Leftrightarrow f(x) \geq 0, \quad \forall x \in K.$$

Of course this definition depends on the specific (functional) nature of the elements of $C(K)$. Equivalently we may write

$$f \geq 0 \Leftrightarrow \exists g \in C(K): \ f = g^* g.$$

The above condition overcomes the specific nature of the elements of $\mathcal{A}$. Also, in $\mathfrak{B}(H)$ we define

$$T \geq 0 \Leftrightarrow < Tx, x > \geq 0, \quad \forall x \in H.$$

It can be shown that

$$T \geq 0 \Leftrightarrow \exists S \in \mathfrak{B}(H): \ T = S^* S.$$

So, we have found a similar condition which describes positivity in both special cases. Thus, it is natural to define positivity on $\mathcal{A}$ as follows:

$$a \geq 0 \Leftrightarrow \exists b \in \mathcal{A}: \ a = b^* b.$$

Of course we have to show that the above order is compatible to the algebraic structure of $\mathcal{A}$. We may define then a linear form to be positive iff

$$a \geq 0 \Rightarrow \varphi(a) \geq 0.$$

Let φ be a linear form in $\mathcal{A}$. We define

$$< a, b > = \varphi(b^* a),$$

while in $C(K)$

$$< f, g > = \varphi(f g^*) = \varphi(g^* f).$$

Here we generalize the second equality in order to avoid an inversion. Again $<,>$ is not generally an inner product, but if

$$\mathcal{J} = \{a | \ a \in A, \ \ < a, a > = 0\},$$

$\mathcal{J}$ is a subspace and $\mathcal{A}/\mathcal{J}$ becomes an inner product space. Then we take the Hilbert space $H = \overline{\mathcal{A}/\mathcal{J}}$. We then define

$$\pi : \mathcal{A} \rightarrow \mathfrak{B}(H) \qquad a \mapsto \pi(a),$$

where

$$\pi(a)[b] = [ab], \qquad \forall b \in \mathcal{A}.$$

$\pi(a)$ is well defined since $\mathcal{I}$ is now a left ideal. Again π is a representation of $\mathcal{A}$ which, generally, is not $1-1$.
With the help of an appropriate construction we can enlarge our Hilbert space in order π becomes $1-1$.

13 Order in C^*-algebras

From now on $\mathcal{A}$ is a C^*-algebra. With the aid of Continuous function calculus we may see an elements of $\mathcal{A}$ as a continuous function. In that way concepts and results on continuous functions apply to $\mathcal{A}$. Also, properties of complex numbers are also transferred in $\mathcal{A}$, since operations are defined pointwisely.

Proposition 13.1: Let $a \in A$ such that $a^*a = aa^*$ and $f \in C(sp(a))$. Then:
(i) (Spectral mapping theorem) $sp(f(a)) = \{f(\lambda) \ : \ \lambda \in sp(a)\}$.
(ii) $a^* = a \Leftrightarrow sp(a) \subseteq \mathbb{R}$.

Proof: We shall use the isomorphism between $C(sp(a))$ and $C^*(a)$, the C^*-algebra generated by a.
(i) Because of the isomorphism, an element is invertible in one algebra iff it is invertible in the other. Therefore, $sp(f(a)) = sp(f)$. But $sp(f) = \{f(\lambda) \ : \ \lambda \in sp(a)\}$, since $t \in sp(f)$ iff $f - t1$ is not invertible in $C(sp(a))$ iff $\exists \lambda \in sp(a) \ : \ (f - t1)(\lambda) = 0 \Leftrightarrow \exists \lambda \in sp(a) \ f(\lambda) = t$.
(ii) Using the isomorphism $id \mapsto a$, $a^* = a \Leftrightarrow id^* = id \Leftrightarrow id(\lambda) \in \mathbb{R}$.$\diamond$

The self-adjoint ($a^* = a$) elements of $\mathcal{A}$ stand for $\mathcal{A}$ the way the elements of $\mathbb{R}$ stand for $\mathbb{C}$. The above proposition shows that the real elements of $\mathcal{A}$ are its normal elements ($a^*a = aa^*$) with real spectrum. We define the positive elements of $\mathcal{A}$ to be its normal elements with positive spectrum. Later we show that this definition is equivalent to the definition we talked about in previous paragraph.

We define an element a of A to be *positive* iff $a^*a = aa^*$ and $sp(a) \subseteq \mathbb{R}^+$. By Proposition 13.1, a is positive iff $a^* = a$ and $sp(a) \subseteq \mathbb{R}^+$.

Hypothesis $a^*a = aa^*$ cannot be avoided by our definition. E.g., the matrix

$$M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

in $\mathbb{C}^{2\times 2}$ has spectrum $\{0\} \subseteq \mathbb{R}^+$, but $M^*M \neq MM^*$.

Now we have to prove that the positivity relation just defined is compatible to the algebraic structure of $\mathcal{A}$.

An *ordered vector space* on $\mathbb{R}$ is a pair $(V, \leq)$, where V is a vector space and $\leq$ is an order in V such that, for each $a, b, c \in V$:
(i) $a \leq b \Rightarrow a + c \leq b + c$.
(ii) $a \leq b$ and $\lambda \geq 0$, then $\lambda a \leq \lambda b$

The *positive cone* V^+ of $(V, \leq)$ is the set

$$V^+ = \{a | \ a \in V \ a \geq 0\}.$$

Clearly,

(i)$a, b \in V^+ \Rightarrow a + b \in V^+$.
(ii) $a \in V^+ \ \wedge \ \lambda \in \mathbb{R}^+ \Rightarrow a + b \in V^+$.
(iii) $a, -a \in V^+ \Rightarrow a = 0$.
(iv) $a \leq b \Leftrightarrow b - a \in V^+$.

Properties (i) and (ii) show that V^+ is also convex. V^+ determines the whole order. Namely, if V is real vector space and V^+ a non-empty subset of V satisfying (i)-(iii),

then $\leq$ is defined through (iv). Then, it is easy to see that $(V, \leq)$ is an ordered vector space and $V^+ = \{a| \ a \in V \ a \geq 0\}$. I.e., an ordered vector space is defined through its positive cone.
It is straightforward that if $a \leq b$ and $\lambda \leq 0$, then $\lambda a \geq \lambda b$. As a special case, $a \leq 0 \Leftrightarrow -a \leq 0$.

Returning to $\mathcal{A}$ we define

$$A^+ = \{a| \ a \in A \ a \geq 0\}.$$

If we set

$$A_h = \{a|a \in A \ a^* = a\},$$

A_h is a real Banach space. If we define then in A_h, $a \leq b \Leftrightarrow b - a \in A^+$, then $(A_h, \leq)$ is an ordered vector space. If $\mathcal{A} = \mathbb{C}$, then $A_h = \mathbb{R}$ and $A^+ = \mathbb{R}^+$. By order in $\mathcal{A}$ we mean the order in A_h, which is not a total order (it is if $\mathcal{A} = \mathbb{C}$), since any two functions are not generally comparable with respect to the order. If $\mathcal{A} = C(K)$, then A_h is a lattice, since for real continuous functions f, g, $f \vee g$, $f \wedge g$ exist and they are continuous. Thus, if $\mathcal{A}$ is commutative, then A_h is a lattice, since $\mathcal{A}$ is then isomorphic to some $C(K)$. The inverse can also be proved i.e., if A_h is a lattice , then $\mathcal{A}$ is commutative.
Someone could expect that $\leq$ would be compatible to the algebraic structure of $\mathcal{A}$ through

$$a, b \in A^+ \Rightarrow ab \in A^+.$$

But, for $a, b \in A^+$, $ab \in A^+ \Leftrightarrow ab = ba$. This makes sense, since the product of self-adjoint elements is self-adjoint iff the elements commute.

Proposition 13.2: If $a \in A$, $a^* = a$ and $\lambda \in \mathbb{R}$, $\lambda \geq ||a||$, then

$$a \geq 0 \Leftrightarrow a - \lambda 1 \leq \lambda.$$

Proof: $(\Rightarrow)$ $||a - \lambda 1|| = sup\{|t| \mid t \in sp(a - \lambda 1)\} = sup\{|t - \lambda| \mid t \in sp(a)\} \leq \lambda$, since if $t \in sp(a)$, then $|t| \leq \lambda$. But $t \geq 0$ therefore, $|t - \lambda| = \lambda - t \leq \lambda$.
$(\Leftarrow)$ We want $sp(a) \subseteq \mathbb{R}^+$. Let $t \in sp(a)$. Then $t - \lambda \in sp(a - \lambda 1)$. Hence, $|t - \lambda| \leq \lambda$. I.e., $-t \leq t - \lambda \leq \lambda$. By the first inequality we get $t \geq 0$.$\diamond$

Proposition 13.3: (i)$a, b \in A^+ \Rightarrow a + b \in A^+$.
(ii) $a \in A^+ \ \wedge \ \lambda \in \mathbb{R}^+ \Rightarrow a + b \in A^+$.
(iii) $a, -a \in V^+ \Rightarrow a = 0$.
(iv) A^+ is closed.

Proof: (i) Obviously, $(a + b)^* = a + b$. Since $a \geq 0$, then applying Proposition 13.2 with $\lambda = ||a||$ we get $||a - ||a||1|| \leq ||a||$. Likewise, $||b - ||b||1|| \leq ||b||$. Therefore, $||a + b - (||a|| + ||b||)1|| \leq ||a - ||a||1|| + ||b - ||b||1|| \leq ||a|| + ||b||$. Thus, with $\lambda = ||a|| + ||b|| \geq ||a + b||$, we get $a + b \geq 0$.
(ii) $(\lambda a)^* = \lambda a$ and $sp(\lambda a) = \{\lambda t| \ t \in sp(a)\} \subseteq \mathbb{R}^+$.
(iii) $a \geq 0 \Rightarrow sp(a) \subseteq \mathbb{R}^+$.
$-a \geq 0 \Rightarrow sp(-a) \subseteq \mathbb{R}^+ \Rightarrow sp(a) \subseteq \mathbb{R}^-$.
Thus, $sp(a) = \{0\}$, therefore $||a|| = sup\{|\lambda| \mid \lambda \in sp(a)\} = 0$ i.e., $a = 0$.
(iv) Let $a_n \in A^+$ and $a_n \to a$. We need to show that $a \in A^+$. By the continuity of $*$, $a^* = a$. Also, $||a - ||a||1|| = lim||a_n - ||a_n||1|| \leq lim||a_n|| = ||a||$. Hence, $a \in A^+$.$\diamond$

Proposition 13.4: Let $a \in A$, $a^* = a$ and $f \in C(sp(a))$. Then,

(i) $f(a) \geq 0 \Leftrightarrow f \geq 0$.
(ii) $-||a||1 \leq a \leq ||a||1$.
(iii) $\exists a^+, a^- \in A$ such that $a = a^+ - a^-$, $a^+, a^- \geq 0$, $a^+a^- = a^-a^+ = 0$ and $||a|| = max\{||a^+||, ||a^-||\}$.

Proof: (i) ($\Rightarrow$) Obviously, $sp(f(a)) = sp(f)$, so $sp(f(a)) \subseteq \mathbb{R}^+$. Therefore, $sp(f(a)) \subseteq \mathbb{R}^+ \Leftrightarrow sp(f) \subseteq \mathbb{R}^+ \Leftrightarrow \{f(\lambda) \mid \lambda \in sp(a)\} \subseteq \mathbb{R}^+ \Leftrightarrow f \geq 0$.
($\Leftarrow$) We need to show that $f(a)^* = f(a)$, if $f \geq 0$. Indeed, $f(a)^* = f^*(a) = f(a)$.
(ii) We need to show $||a||1 \pm a \geq 0$. Let $f_\pm(t) = ||a|| \pm t$, $t \in sp(a)$. Obviously, $f_\pm \geq 0$. Hence, $||a||1 \pm a = f_\pm(a) \geq 0$.
(iii) $a = id(a)$. Since $a^* = a$, $sp(a) \subseteq \mathbb{R}$ and so id is a real function, hence $id = id^+ - id-$, $id^+, id^- \geq 0$, $id^+id^- = id^-id^+ = 0$ and $||id|| = max\{||id^+||, ||id^-||\}$. We set $a^+ = id^+$ and $a^- = id^-$ and we get the desired properties.$\diamond$

Let a in A. We write

$$a = \frac{a + a^*}{2} + i\frac{a - a^*}{2i} = Rea + iIma,$$

where $\frac{a+a^*}{2}$ is the real part of a and $\frac{a-a^*}{2i}$ the complex part of a. Also, $(Rea)^* = Rea$ and $(Ima)^* = Ima$ and we get directly that the analysis $a = u + iv$, where $u^* = u$, $v^* = v$ is unique. Also, $u = u^+ - u^-$, $v = v^+ - v^-$ therefore, $a = u + iv = u^+ - u^- + iv^+ - iv^-$ i.e., each element a is written a linear combination of four elements of A^+.

We prove now the equivalence between the two definitions of positivity.

Proposition 13.5: Let a in A. The following are equivalent:
(i) $a \geq 0$.
(ii) $\exists h \geq 0 \ : \ h^2 = a$.
(iii) $\exists b \in A \ : \ b^*b = a$.

Proof: $((i) \Rightarrow (ii))$ We consider the function

$$f(t) = t^{\frac{1}{2}}, \quad t \in sp(a).$$

f is well defined since $sp(a) \subseteq \mathbb{R}^+$. We set $h = f(a)$. Then, $h^2 = a$ and since $f \geq 0$, then by Proposition 13.4(i), $h = f(a) \geq 0$.
$((ii) \Rightarrow (iii))$ We take $b = h$.
$((ii) \Rightarrow (i))$ Obviously, $a^* = a$. If $sp(h) \subseteq \mathbb{R}$, then $sp(a) = \{\lambda^2 | \ \lambda \in sp(h)\} \subseteq \mathbb{R}^+$. We have used only that $h^* = h$ i.e., not the fact that $h \geq 0$. This will be used later.

The proof of the direction $((iii) \Rightarrow (i))$ is rather technical and we shall use the following two propositions as lemmas.

Proposition 13.6: Let a in A and $-a^*a \geq 0$, then $a = 0$.

Proof: It suffices to show that $a^*a \geq 0$. We set $a = u + iv$, $u^* = u$, $v^* = v$. Then,

$$a^*a = u^2 + v^2 + i(uv - vu),$$

$$aa^* = u^2 + v^2 + i(vu - uv).$$

Hence, $a^*a = 2u^2 + 2v^2 + (-a^*a)$, since $2u^2, 2v^2 \geq 0$, because $(-aa^*)^* = -aa^*$ and by Proposition 7.1

$$sp(-aa^*) \cup \{0\} = sp(-a^*a) \cup \{0\} \subseteq \mathbb{R}^+. \quad \diamond$$

Proposition 13.7: If a is in A and there is $k \in \mathbb{N}$ such that $a^k = 0$ i.e., a is *nilpotent*, and $a^*a = aa^*$, then $a = 0$.

Proof: If $n \geq k$, then $a^n = 0$. Since a is normal

$$||a|| = r(a) = lim_{n\to\infty}||a^n||^{\frac{1}{n}} = 0.$$

Therefore, $a = 0.\diamond$

Proof of $((iii) \Rightarrow (i))$ **of Proposition 13.5:** Let $a = b^*b$. Obviously $a^* = a$. We may write $a = a^+ - a^-$. It suffices to prove that $a^- = 0$. We set $c = ba^-$. Then, $c^*c = a^-b^*ba^- = a^-(a^+ - a^-)a^- = -(a^-)^3$. Since $a^- \geq 0$, $(a^-)^3 \geq 0$. Hence, $-c^*c = (a^-)^3 \geq 0$, thus, by Proposition 13.6, $c = 0$. So, $(a^-)^3 = 0$ and by Proposition 13.7, $a^- = 0.\diamond$

If $a \in A^+$, then $a^{\frac{1}{2}}$ is called the *square root* of a, where

$$a^{\frac{1}{2}} = f(a), \quad f(t) = t^{\frac{1}{2}}, \quad t \in sp(a).$$

Obviously, $(a^{\frac{1}{2}})^2 = a$.

Proposition 13.8: Let H be a Hilbert space and $T \in \mathfrak{B}(H)$. Then

$$T \geq 0 \Leftrightarrow < Tx, x > \geq 0, \qquad \forall x \in H,$$

where $T \geq 0$ refers to the general definition of order in C^*-algebras.

We need the following lemma for the proof of this proposition.

Proposition 13.9: $T = 0 \Leftrightarrow < Tx, x > = 0, \qquad \forall x \in H.$

Proof: By polarization identity,

$$< Tx, y > = \frac{1}{4}\sum_{n=0}^{3} i^n < T(x + i^n y), x + i^n y > = 0, \qquad \forall x, y \in H.$$

Therefore, $T = 0.\diamond$

Direct consequences of Proposition 13.9 are the following:

$$S = T \Leftrightarrow < Sx, x > = < Tx, x >, \quad \forall x \in H.$$

$$T^* = T \Leftrightarrow < Tx, x > \in \mathbb{R}, \quad \forall x \in H.$$

Proof of Proposition 13.8: $(\Rightarrow)$ Since there is S in $\mathfrak{B}(H)$ such that $T = S^*S$, $< Tx, x > = < S^*Sx, x > = < Sx, Sx > \geq 0$.
$(\Leftarrow)$ Since $< Tx, x > \in \mathbb{R}$, for each x in H, then, by Proposition 13.9, $T^* = T$. We may write $T = T^+ - T^-$. We need to show that $T^- = 0$.
Since for each y in H $0 \leq < Ty, y >$, then

$$0 \leq < T(T^-x), T^-x > = < (T^+ - T^-)T^-x, T^-x >$$

$$= - < (T^-)^2x, T^-x > = - < (T^-)^3x, x > \leq 0,$$

since $T^- \geq 0$. Thus, by Proposition 13.9, $(T^-)^3 = 0$ i.e., $T = 0.\diamond$

14 The Gelfand-Naimark-Segal construction

A linear form φ in $\mathcal{A}$ is called *positive* iff

$$a \geq 0 \Rightarrow \varphi(a) \geq 0.$$

Equivalently, due to linearity, iff φ preserves the order.

In $C(K)$ the positive linear forms "are" the regular measures on K. E.g., if $x \in K$, the positive linear form

$$\delta_x(f) = f(x)$$

corresponds to the Dirac measure with respect to x.
If H is a Hilbert space and $x \in H$, then

$$\omega_x(T) = < Tx, x >$$

is a positive linear form in $\mathfrak{B}(H)$.

If φ is a positive linear form in $\mathcal{A}$, we define

$$< a, b > = \varphi(b^* a).$$

Proposition 14.1: (I)$<,>$ is a "semi-inner product" i.e.,
(i) It is a linear form with respect to the first variable while it is anti-linear with respect to the second.
(ii) $< a, a > \geq 0, \ \forall a \in A$ i.e., $\varphi(a^* a) \geq 0$.
(iii) $< a, b > = \overline{< b, a >}, \ \forall a, b \in A$ i.e., $\varphi(b^* a) = \overline{\varphi(a^* b)}$. Especially, $\varphi(a^*) = \overline{\varphi(a)}$.
(II) (Cauchy-Schwarz inequality) $| < a, b > |^2 \leq < a, a >< b, b >, \ \forall a, b \in A$ i.e., $|\varphi(b^* a)|^2 \leq \varphi(a^* a)\varphi(b^* b)$.
(III) φ is continuous with $||\varphi|| = \varphi(1)$.

Proof: (I)(i) is trivial.
(ii) Since $a^* a \geq 0$, then $< a, a > = \varphi(a^* a) \geq 0$.
(iii) Using only (i) we can prove the "polarization identity"

$$< a, b > = \sum_{n=0}^{3} i^n < x + i^n y, x + i^n y >.$$

Using $< a, a > \in \mathbb{R}$, after simple calculations we reach $< a, b > = \overline{< b, a >}$.
(II) Obviously,

$$0 \leq < a + \lambda b, a + \lambda b > = |\lambda|^2 < b, b > + 2Re(\overline{\lambda} < a, b >) + < a, a >. \quad (*)$$

Case (1): $< b, b > \neq 0$. In $(*)$ we set

$$\lambda = -\frac{a, b >}{< b, b >},$$

and we get the inequality (II).
Case (2): $< a, a > \neq 0$. We interchange a, b and we use Case (1).
Case (3): $< a, a > = < b, b > = 0$. Then $(*)$ becomes $2Re(\overline{\lambda} < a, b >) \geq 0$. We set $\lambda = - < a, b >$. Then, $-2| < a, b > |^2 \geq 0$ therefore, $| < a, b > |^2 = 0 \leq < a, a >< b, b >$.

(III) $||\varphi|| = \varphi(1)$ is expected, since in case our algebra is $C(K)$, if we see φ as a measure, then

$$|\varphi(f)| = |\int f d\mu| \leq ||f|| \mu(K)$$

and $\varphi(1) = \int d\mu = \mu(K)$. Hence, $||\varphi|| = \varphi(1)$.
By Cauchy-Schwarz inequality,

$$|\varphi(a)|^2 = |\varphi(1^* a)|^2 \leq \varphi(1^* 1)\varphi(a^* a) = \varphi(1)\varphi(a^* a).$$

Therefore, since $|\varphi(a)|^2 \leq \varphi(1)^2 ||a||^2$, it suffices to show

$$\varphi(a^* a) \leq \varphi(1)||a^* a||$$

i.e., it suffices to show that φ is bounded for self-adjoint a. I.e., we need to show $\varphi(a) \leq \varphi(1)||a||$, for each $a^* = a$. Then, $\varphi(a) \in \mathbb{R}$, since $\varphi(a) = \varphi(a^*)\overline{\varphi(a)}$, therefore, we need $-\varphi(1)||a|| \leq \varphi(a) \leq \varphi(1)||a||$ i.e., $\varphi(-||a||1) \leq \varphi(a) \leq \varphi||a||(1)$, which holds, since $-||a||1| \leq a \leq ||a||1|$, by Proposition 13.4(ii).$\diamond$

We set

$$\mathcal{J} = \{a | \ a \in A, \ \varphi(a^* a) = 0\}.$$

By Cauchy-Schwarz inequality we get

$$\mathcal{J} = \{a | \ a \in A, \ \varphi(b^* a) = 0, \ \forall b \in A\}.$$

Proposition 14.2: $\mathcal{J}$ is a closed (with respect the $||.||$-topology) left ideal.

Proof: $\mathcal{J}$ is closed as the inverse image of $\{0\}$ under the continuous function $a \mapsto \varphi(a^* a)$.
Obvious calculations show that $\mathcal{J}$ is a subspace. It is also a left ideal i.e.,

$$a \in \mathcal{I}, \ b \in A \Rightarrow ba \in \mathcal{J},$$

since $\varphi((ba)^* ba) = \varphi(a^* b^* ba) = \varphi((b^* ba)^* a) = 0.\diamond$

It is easy to see that in the vector space $\mathcal{A}/\mathcal{J}$ the operation

$$< [a], [b] > = < a, b >$$

is well-defined and an inner product. We set $H = \mathcal{A}/\mathcal{J}$. If $a \in A$, we define the operation

$$\pi_0(a) : \mathcal{A}/\mathcal{J} \to \mathcal{A}/\mathcal{J}$$

$$\pi_0(a)[b] = [ab].$$

Proposition 14.3: The operation $\pi_0(a)$ is well-defined, linear and continuous. Therefore, $\pi_0(a)$ has a unique extension:

$$\pi(a) : H \to H.$$

In that way we define

$$\pi : \mathcal{A} \to \mathfrak{B}(H)$$

$$a \mapsto \pi(a).$$

Mapping π is a representation of $\mathcal{A}$.

Proof: $\pi_0(a)$ is well-defined: $b \sim c \Rightarrow b - c \in \mathcal{J} \Rightarrow a(b-c) \in \mathcal{J}$, by Proposition 14.2, and hence, $ab \sim ac$.
Linearity of $\pi_0(a)$ is trivial.
$\pi_0(a)$ is bounded: We show that

$$||\pi_0(a)|| \leq ||a||.$$

I.e., for each $b \in A$

$$||\pi_0(a)b|| \leq ||a||||b||.$$

I.e., we want

$$||[ab]|| \leq ||a||||b|| \Leftrightarrow < ab, ab > \leq < b, b > ||a||^2$$

$$\Leftrightarrow \varphi((ab)^*ab) \leq \varphi(b^*b)||a||^2 \Leftrightarrow \varphi(b^*(a^*a)b) \leq \varphi(b^*b)||a^*a||.$$

This expression suggests that we have to set

$$\varphi_b(c)\varphi(b^*cb).$$

φ_c is a positive linear form ($\varphi_c(a^*a) = \varphi((ab)^*ab) \geq 0$), therefore, $||\varphi_c|| = \varphi_c(1)$ i.e., $\varphi(b^*(a^*a)b) \leq \varphi(b^*b)||a^*a||$, which is exactly what we wanted to show.
Obviously, $\pi(a) \in \mathfrak{B}(H)$ with

$$||\pi(a)|| = ||\pi_0(a)|| \leq ||a||.$$

Clearly, π is linear, multiplicative and $\pi(1) = I$. Also, it is continuous, since $||\pi(a)|| \leq ||a||$.
Also

$$\pi(a^*) = \pi(a)^*,$$

since for each $b, c \in A$

$$< \pi(a)^*[b], [c] > = < [b], \pi(a)[c] > = < [b], [ac] > = \varphi((ac)^*b)$$

$$= \varphi(c^*(a^*b)) = < [a^*b], [c] > = < \pi(a^*)[b], [c] >.$$

I.e., $\pi(a^*), \pi(a)^*$ are equal on the dense subspace $\mathcal{A}/\mathcal{J}$, therefore they are equal.$\diamond$

The general situation that we study here is the following: We have a C^*-algebra $\mathcal{A}$ and a positive linear form φ on $\mathcal{A}$. As a special case $\mathcal{A}$ can be $\mathfrak{B}(H)$, where H is a Hilbert space and positive linear form ω_ξ, where $\xi \in H$ and

$$\omega_\xi(T) = < T\xi, \xi >.$$

The construction we described above sends $\mathcal{A}$ to some $\mathfrak{B}(H)$, but not only that. As we show in the next proposition, it sends φ to some ω_ξ, and in a sense this representation is essentially unique.

Proposition 14.4: If $\mathcal{A}$ is a C^*-algebra and φ is a positive linear form on $\mathcal{A}$, then there is a Hilbert space H, a vector $\xi \in H$ and a representation $\pi : \mathcal{A} \to \mathfrak{B}(H)$ such that:
(i) The set $\{\pi(a)\xi | \ a \in A\}$ is dense in H. Then, ξ is called a *cyclic vector* for π and a

representation with a cyclic vector is called *cyclic*.
(ii) For each a in A, $\varphi(a) = \omega_\xi(\pi(a))$ i.e., π also "preserves" φ.
(iii) If there are $H^{'}, \xi^{'}, \pi^{'}$ with the same properties, then $\pi, \pi^{'}$ are unitarily equivalent i.e., there is isomorphism $U : H \to H^{'}$ of Hilbert spaces (i.e., linear isometry, onto $H^{'}$) such that for each a in A

$$\pi^{'}(a) = U\pi(a)U^{-1}.$$

Proof: Let H, ξ, π the Hilbert space and the previously constructed representation. We set $\xi = [1]$. Then:
(i) $\pi(a)\xi = [a1] = [a]$. Hence, $\{\pi(a)\xi |\ a \in A\} = \{[a] |\ a \in A\} = \mathcal{A}/\mathcal{J}$, which is dense in H.
(ii) $\omega_\xi(\pi(a)) = <\pi(a)\xi, \xi> = <[a], [1]> = \varphi(1^*a) = \varphi(a)$.
(iii) Let $H^{'}, \xi^{'}, \pi^{'}$ with the same properties. We need to find an isomorphism $U : H \to H^{'}$. The most natural mapping is the one which sends $\pi(a)\xi$ (i.e., $[a]$) to $\pi(a)^{'}\xi^{'}$. Then we define:

$$U_0 : \mathcal{A}/\mathcal{J} \to H^{'}$$

$$U_0[a] = \pi^{'}(a)\xi^{'}.$$

We show that U_0 is well defined by showing that it is an isometry.

$$||\pi^{'}(a)\xi^{'}||^2 = <\pi^{'}(a)\xi^{'}, \pi^{'}(a)\xi^{'}> = <\pi^{'}(a)^*\pi^{'}(a)\xi^{'}, \xi^{'}> =$$
$$<\pi^{'}(a^*a)\xi^{'}, \xi^{'}> = \varphi(a^*a) = <\pi(a^*a\xi, \xi> = ||\pi(a)\xi||^2.$$

U_0 is well defined:

$$a \sim b \Rightarrow ||\pi^{'}(a)\xi^{'} - \pi^{'}(b)\xi^{'}|| = ||\pi^{'}(a-b)\xi^{'}|| = ||[a-b]|| = 0.$$

Therefore, $\pi^{'}(a)\xi^{'} = \pi^{'}(b)\xi^{'}$.
Clearly U_0 is linear. Hence, since

$$U_0(\mathcal{A}/\mathcal{J}) = \{\pi^{'}(a)\xi^{'} |\ a \in A\}$$

is dense in $H^{'}$, U_0 is extended to $U : H \to H^{'}$, linear isometry, onto $H^{'}$.
In order to show that $\pi^{'}(a) = U\pi(a)U^{-1}$, it suffices to show that for each $b \in A$

$$U^{-1}\pi^{'}(a)U[b] = \pi(a)[b] \Leftrightarrow U^{-1}\pi^{'}(a)\pi^{'}\xi = [ab]$$

$$\Leftrightarrow U^{-1}\pi^{'}(ab)\xi = [ab] \Leftrightarrow U[ab] = \pi^{'}(ab)\xi,$$

which holds.$\diamond$

The above procedure of generation of a representation of $\mathcal{A}$ given a positive linear form on $\mathcal{A}$ is called *Gelfand-Naimark-Segal (GNS)construction.*

15 The general Gelfand-Naimark theorem

The representation we described in previous paragraph is not generally $1-1$. I.e., it is possible to have $a \neq 0$ and $\pi(a) = 0$. But if $\varphi(a^*a) > 0$, then $[a] \neq 0$, therefore $\pi(a)[1] \neq 0$ i.e., $\pi(a) \neq 0$. We shall prove that for each $a \neq 0$ we can find such an φ. The corresponding representation will have $\pi(a) \neq 0$. "Summing" all Hilbert spaces so constructed and the corresponding representations, we find a faithful representation.

Proposition 15.1: For each a in A, if $a \neq 0$, then there exists a positive linear form φ such that

$$\varphi(a^*a) > 0.$$

Proof: We consider the real Banach space A_h. A^+ is a closed and convex subset of A_h. If $a \neq 0$, then $a^*a \in A^+$, hence, $-a^*a \notin A^+$. By the second geometric form of Hahn-Banach theorem there is linear $f : A_h \to \mathbb{R}$ and there is $x \in \mathbb{R}$ such that

$$f(-a^*a) < x < f(b), \quad \forall b \in A^+,$$

where $x < f(0) = 0$. f is positive i.e., $b \geq 0 \Rightarrow f(b) \geq 0$, since if $f(b) < 0$, then for $\lambda = -\frac{x}{f(b)} > 0$, we get $\lambda b \in A^+$ and $f(\lambda b) = x$, which is absurd. Also, $f(-a^*a) < x < 0$, hence $f(a^*a) > 0$.
If we define $\varphi : A \to \mathbb{C}$

$$\varphi(u + iv) = f(u) + if(v),$$

then φ is linear and positive, since $b \in A^+ \Rightarrow \varphi(b) = f(b) \geq 0$. Also, $\varphi(a^*a) = f(a^*a) > 0$.$\diamond$

Proposition 15.2 (general Gelfand-Naimark theorem, 1943): Each C^*-algebra has a faithful representation in a Hilbert space.

Proof: For each $a \neq 0$ in A we choose φ_a positive linear form such that

$$\varphi_a(a^*a) > 0.$$

Then a Hilbert space H_α is constructed and a representation $\pi_a : \mathcal{A} \to \mathfrak{B}(H_a)$. Let $A_0 = A - \{0\}$. We set

$$H = \bigoplus_{a \in A_0} H_a,$$

the direct sum of the Hilbert spaces H_a (see Appendix for its definition). We define

$$\pi : \mathcal{A} \to \mathfrak{B}(H),$$

$$\pi(x) = \oplus_{a \in A_0} \pi_a(x).$$

$\pi_a(x)$ is a bounded linear operator $H_a \to H_a$ with $||\pi_a(x)|| \leq ||x||$. Then, as we show in the Appendix, $\pi(x)$ is a bounded linear operator $H \to H$, with $||\pi(x)|| \leq ||x||$.
Clearly, π is linear, multiplicative and $\pi(1) = I$. Also $\pi(x^*) = \pi(x)^*$, since

$$\pi(x^*) = \oplus_{a \in A_0} \pi_a(x^*) = \oplus_{a \in A_0} \pi_a(x)^* = \pi(x)^*.$$

The last equality holds since, if we set $T_a = \pi_a(x), T = \pi(x)$, then

$$T = \oplus_{a \in A_0} T_a \Rightarrow T^* = \oplus_{a \in A_0} T_a^*.$$

Finally, π is $1-1$, since if $a \neq 0$, then $\varphi_x(x^*x) > 0$. Hence $[x]_x \neq 0$, $[x]_x \in H_x$, therefore, $\pi_x(x)[1]_x = [x]_x \neq 0$ i.e., $\pi_x(x) \neq 0$, so $\pi(x) = \oplus_{a \in A_0} \pi_a(x) \neq 0$.$\diamond$

Especially π is an isometry too, due to the following proposition.

Proposition 15.3: If $\mathcal{A}, \mathcal{B}$ are C^*-algebras and $\pi : \mathcal{A} \to \mathcal{B}$ an algebraic structure preserving mapping, then:
(i) π is continuous and $||\pi|| = 1$.
(ii) If π is $1-1$, then it is an isometry.

Proof: (i) Since $\pi(1) = 1$, it suffices to show that $||\pi(x)|| \leq ||x||$. It is easy to see that if an element is invertible in $\mathcal{A}$, then its image is invertible in $\mathcal{B}$. Then, $sp(\pi(x)) \subseteq sp(x)$. Thus,

$$||\pi(x)||^2 = ||\pi(x^*x)|| = sup\{|\lambda| \ \lambda \in sp(\pi(x^*x))\}$$
$$\leq sup\{|\lambda| \ \lambda \in sp(x^*x)\} = ||x^*x|| = ||x||^2.$$

For the proof of (ii) we need the following proposition.

Proposition 15.4: If a is in A, $x^* = x$ and $f \in C(sp(x))$, then $\pi(f(x)) = f(\pi(x))$.

Proof: If f is a polynomial, then the equality is obvious. Since $x^* = x$, polynomials are dense in $C(sp(x))$. Let (p_n) a sequence of polynomials such that $p_n \to f$. Then, $\pi(f(x)) = \pi(lim p_n(x)) = lim \pi(p_n(x)) = lim p_n(\pi(x)) = f(\pi(x))$.$\diamond$

Proof of (ii) of Proposition 15.3: It suffices to show that the only inequality which appears in the sequence of equalities in the proof of (i), it is also an equality. I.e., it suffices to show that $sp(\pi(x)) = sp(x)$, knowing that $sp(\pi(x)) \subseteq sp(x)$. Suppose that $sp(\pi(x)) \neq sp(x)$. Then, there is an $f \neq 0$ in $C(sp(x))$ such that $f|_{sp(\pi(x))} = 0$ and $f \neq 0 \Rightarrow f(x) \neq 0 \Rightarrow \pi(f(x)) \neq 0$. But, $f|_{sp(\pi(x))} = 0 \Rightarrow f(\pi(x)) = 0$, which is absurd, by Proposition 15.4.$\diamond$

The defect of the above method is that the constructed Hilbert space H is too large. E.g., as an uncountable sum of non-trivial Hilbert spaces it is always non-separable. Also, the constructed π often is not the natural representation which may fit $\mathcal{A}$. E.g., if $\mathcal{A} = \mathfrak{B}(H)$, π is not the identity! In some cases the size of the representation π can be reduced. E.g., if if we have a faithful representation it suffices to sum the representations π_i which derive from a family of positive linear forms $(\varphi_i)_{i \in I}$ such that

$$\forall x \in A \ x \neq 0 \ \exists i \in I : \ \varphi_i(x^*x) > 0.$$

Then, the method of the proof of Proposition 15.2 applies and gives the desired "faithfulness". Thus, if there is φ such that $\varphi(x^*x) > 0$, $\forall x \neq 0$, then φ itself suffices. E.g., if $\mathcal{A} = C([0,1])$ form φ which corresponds to Lebesgue measure has this property. The corresponding Hilbert space is, of course, $L^2([0,1])$ and we get our initial example.

16 Appendix: the direct product of Hilbert spaces

Let $(H_i)_{i\in I}$ a family of Hilbert spaces. The *algebraic direct sum* of H_i is the set

$$H_0 = \{(x_i)_{i\in I} \mid x_i \in H_i \ \wedge \ \{i \in I \ : \ x_i \neq 0\} \ is \ finite\},$$

equipped with addition and multiplication by complex number defined pointwisely and with the following inner product:

$$< x, y > = \sum_{i\in I} < x_i, y_i >,$$

where $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$. It is easy to see that $<, >$ is well defined and H_0 is an inner product space.
The *(Hilbert) direct sum* of H_i is the Hilbert space

$$\bigoplus_{i\in I} H_i = \widetilde{H_0}.$$

Let $T_i : H_i \to H_i$ be linear operators. We set $T_0 : H_0 \to H_0$ with

$$T_0(x_i)_{i\in I} = (T_i x_i)_{i\in I}.$$

Obviously, T_0 is a linear operator. If T_i are bounded with $sup||T_i|| < \infty$, then

$$||T_0(x_i)_{i\in I}||^2 = \sum_{i\in I} ||T_i x_i||^2 \leq (sup_{i\in I}||T_i||)^2 \sum_{i\in I} ||x_i||^2$$

$$= (sup_{i\in I}||T_i||)^2 ||(x_i)_{i\in I}||^2.$$

I.e., T_0 is bounded with

$$||T_0|| \leq sup_{i\in I}||T_i||$$

Hence, T_0 is extended uniquely to the bounded linear operator

$$\oplus_{i\in I} T_i : \bigoplus_{i\in I} H_i \to \bigoplus_{i\in I} H_i,$$

$$|| \oplus_{i\in I} T_i|| = ||T_0|| \leq sup_{i\in I}||T_i||.$$

Finally we show that

$$\oplus_{i\in I}(T_i^*) = (\oplus_{i\in I} T_i)^*.$$

It suffices to show that these operators are identical on the dense subspace H_0. If $(x_i)_{i\in I}, (y_i)_{i\in I} \in H_0$, then

$$< \oplus_{i\in I}(T_i^*)(x_i)_{i\in I}, (y_i)_{i\in I} > = < (T_i^* x_i)_{i\in I}, (y_i)_{i\in I} >$$

$$= \sum_{i\in I} < T_i^* x_i, y_i > = \sum_{i\in I} < x_i, T_i y_i >$$

$$= < (x_i)_{i\in I}, (T_i y_i)_{i\in I} > = < (x_i)_{i\in I}, (\oplus_{i\in I} T_i)(y_i)_{i\in I} >$$

$$= < (\oplus_{i\in I} T_i)^*(x_i)_{i\in I}, (y_i)_{i\in I} >.$$

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